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# Geometrical interpretation of D-branes in gauged WZW models

**Juan Maldacena*and Nathan Seiberg**

*School of Natural Sciences, Institute for Advanced Study*
*Einstein Drive, Princeton, New Jersey, 08540, USA*
*E-mail:* malda@bose.harvard.edu, seiberg@ias.edu

**Gregory Moore**

*Department of Physics, Rutgers University*
*Piscataway, New Jersey, 08855-0849, USA*
*E-mail:* gmoore@physics.rutgers.edu

Abstract: We show that one can construct D-branes in parafermionic and WZW theories (and their orbifolds) which have very natural geometrical interpretations, and yet are not automatically included in the standard Cardy construction of D-branes in rational conformal field theory. The relation between these theories and their T-dual description leads to an analogy between these D-branes and the familiar A-branes and B-branes of $N = 2$ theories.

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*Department of Physics, Harvard University, Cambridge, MA 02138, USA.

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## Contents

# 1. Introduction

D-branes are an essential element of modern string theory [1]. Therefore, given an on-shell closed string background a fundamental question is: "What are the D-branes in the given background?" This innocent question turns out to be a difficult and subtle problem. A useful approach in answering this question is to phrase it in the language of boundary states in abstract conformal field theory. General sewing conditions for open string boundary conditions are well-understood mostly in rational conformal field theory — RCFT (for a partial list of references see, e.g. [2]–[10]). Moreover, it seems clear that one should also impose local, conformally invariant boundary conditions. Combining these conditions in a useful and general way is an outstanding unsolved problem.

With this motivation in mind (as well as others) many authors have studied D-branes in the solvable WZW models for compact Lie group $G$ using the exact solution of the conformal field theory. A beautiful picture has begun to emerge in which, among other things, D0-branes blow up into branes wrapping conjugacy classes in the group [11]–[14]. The next interesting case to study is that of gauged WZW models, and the present paper addresses the geometrical interpretation of branes in those models in the case where $G$ is SU(2).

The subgroups of $G = \mathrm{SU}(2)$ which we can gauge include the $ADE$ discrete subgroups, the U(1) subgroup, and $G$ itself. These groups can act on the left or the right. The $G/G$ model is a topological theory which has been studied, for example, in [15]. As noted in [16] this model is the basis for a physical explanation of the recent mathematical result of [17]. Indeed, the branes very naturally correspond to a basis for the Verlinde algebra. Other gaugings will be the focus of the present paper. Indeed, we will focus on the $\mathrm{SU}(2)/\mathrm{U}(1)$ model, where we gauge a vector (or axial) U(1) and on the $\mathrm{SU}(2)/\mathbb{Z}_n$ models where we gauge a left action of $\mathbb{Z}_n$. The generalization to other discrete subgroup actions is left as an interesting open problem.

The novelty of the results below is that even in these well-studied RCFT's it is possible to go beyond the D-branes constructed via the standard Cardy theory while still maintaining a good geometrical picture. This is not to say that the branes were completely unknown. A general theory of symmetry breaking branes was set forth in [6, 7] and we believe the branes constructed below fit into that theory. Moreover, D-branes in parafermionic models were studied in [8] from an algebraic perspective. The emphasis in the present paper is on the *geometrical* interpretation of the theory.

Geometrically, the $\mathrm{SU}(2)/\mathrm{U}(1)$ parafermion theory is a sigma model with a disk target space. In this paper we show that the simple D-branes are D0-branes at special points at the boundary of the disk together with D1-branes stretched between these points. One of the interesting aspects of these D-branes is the interplay between the exact algebraic description of the branes and their geometric description in the target space.

In bosonic theories the D-branes are not oriented. In supersymmetric theories the D-branes can be oriented. In the SU(2) theory the D0 and D2-branes become orientable and they each have a moduli space $S^3$ corresponding to translations around the group. The supersymmetric $\mathrm{SU}(2)/\mathrm{U}(1)$ theory has D1-branes and D2-branes. Some of them are orientable.

One of our main results is that in addition to these branes which we refer to as A-branes these theories also have other branes. We refer to the new branes as B-branes. This terminology is borrowed from $\mathcal{N}=2$ theories [18] and is being extended here. The B-branes can be found by the following procedure. These theories are T-dual to their $\mathbb{Z}_k$ orbifolds. The A-branes in the covering theory lead, using the method of images, to some branes in the orbifold theory. The T-dual version of these brane are not the original A-branes in the model we started with. These are our new B-branes. The construction seems to fit into the general scheme proposed in [6, 7] for analyzing symmetry breaking boundary conditions.

The B-branes in the SU(2) theory are like D1-branes wrapping around the group. Several such D1-branes blow up into a D3-brane. The moduli space of these branes is $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)/\mathrm{U}(1)^2$.

The B-branes in the parafermion theory are a D0-brane and D2-branes at the center of the disk and most of them are unstable. Of particular interest is the case of $k$ even where a bound state of $\frac{k}{2}+1$ such D0-branes at the center of the disk is two separate D2-branes. Each of them covers the whole disk and is stable.

In section 2 we review some well known results about conformal field theories whose chiral algebras are based on U(1) and SU(2) affine Lie algebras and the GKO $\mathrm{SU}(2)/\mathrm{U}(1)$ coset theory. Here we discuss only the closed strings. In section 3 we

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start by considering the D-branes in the U(1) theory. This elementary example is useful since it demonstrates our method in a widely known context. We then turn to the D-branes in the parafermion theory. We discuss the A-branes and the B-branes and study their properties. In particular, we compute the spectrum of open strings living on the various branes or stretched between two different branes. In section 4 we present two effective descriptions of these B-branes. The first is as a bound state of D0-branes (as in [11, 19]) and the second is an effective theory of a D2-brane, as in [12]. Section 5 is devoted to the superparafermion theory and its D-branes and builds on the results of [8]. The A-branes are found to be oriented D1-branes. Most of the B-branes are unoriented D2-branes but a few of them are oriented D2-branes. In section 6 we consider the $SU(2)_k$ theory and its $\mathbb{Z}_{k_1}$ orbifolds (Lens spaces). In addition to the known A-branes we also discuss the new B-branes and explore some of their properties.

In several appendices we discuss some more technical details. In appendix A we list the U(1) and SU(2) characters and their modular properties. In appendix B we compute the open string spectrum between A-branes and B-branes in the parafermion theory. Appendix C is devoted to three explicit examples of parafermion theories which exhibit special features. In appendix D we present a calculation of the shape of the various branes which substantiates our geometrical pictures. Appendix E is a review of $\mathcal{N} = 2$ theories and appendix F is an explicit analysis of the first superparafermion theory.

In addition to the intrinsic interest in these models as solvable conformal field theories, we would like to mention a few other applications of these theories and their D-branes:

(1) The SU(2) theory arises in string theory in the background of NS5-branes and in condensed matter theory in the Kondo problem as well as in other places.

(2) The SU(2) theory is closely related to SL(2) which appears in string theory in the contexts of two and three-dimensional black holes.

(3) The $SU(2)_k/\mathbb{Z}_{k_1}$ theory arises when $k/k_1$ NS5-branes and $k_1$ Kaluza-Klein monopoles coincide.

(4) Parafermions appear in many string constructions. In particular, they are closely related to the two-dimensional black hole, both the Lorentzian and euclidean (cigar) versions.

(5) The $\mathcal{N} = 2$ super-discrete series is used as a building block in Calabi-Yau compactifications. Some of the B-branes constructed in the $\mathcal{N} = 2$ minimal models might be very interesting in understanding further the results of Brunner and Distler [20] on torsion branes.

We hope that our techniques will be useful in the study of these models.



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## 2. Review of closed strings in parafermion theory

### 2.1 Algebraic considerations

$\mathrm{U}(1)_k$ The $\mathrm{U}(1)_k$ chiral algebra with $k \in \mathbb{Z}$ extends the chiral algebra generated by a gaussian $\mathrm{U}(1)$ current $J = i\sqrt{2k}\partial X$ by including charged fields of dimension $k$ and charge $\pm 2k$, $\exp[\pm i\sqrt{2k}X(z)]$. The representations of $\mathrm{U}(1)_k$ are labeled by an integer $n$, defined modulo $2k$, and identified with the charge. The lowest $L_0$ eigenvalue in the representation labeled by $n$ is

$$\Delta_n = \frac{n^2}{4k}, \tag{2.1}$$

where we choose the fundamental domain $n = -k+1, -k+2, \ldots, k$. The character of this representation is

$$\mathrm{Tr}_{\mathcal{H}_n} q^{L_0 - 1/24} e^{2\pi i z J_0} = \frac{\Theta_{n,k}(\tau, 2z)}{\eta(\tau)}. \tag{2.2}$$

(See appendix A for conventions on theta functions.) We often specialize to $z = 0$ and define

$$\psi_n(q) = \frac{1}{\eta(q)} \sum_{r \in \mathbb{Z} + \frac{n}{2k}} q^{kr^2}. \tag{2.3}$$

The fusion rules are $n_1 \times n_2 = (n_1 + n_2) \mod 2k$. The action of the modular transformation $S$ on the characters is

$$\psi_n(q') = \frac{1}{\sqrt{2k}} \sum_{n'} e^{-i\pi n n'/k} \psi_{n'}(q), \qquad q = e^{2\pi i \tau}, \qquad \tau' = -\frac{1}{\tau}. \tag{2.4}$$

The diagonal modular invariant made of these representations is

$$Z_{\mathrm{diag}} = \sum_n |\psi_n(q)|^2 \tag{2.5}$$

and it describes a boson on a circle of radius $\sqrt{2k}$ (in units where $\alpha' = 2$). The most general modular invariant is

$$Z_l = \sum_{\substack{n+\overline{n}=0 \mod 2l \\ n-\overline{n}=0 \mod 2l'}} \psi_n(q) \psi_{\overline{n}}(\overline{q}), \qquad ll' = k, \qquad l, l' \in \mathbb{Z}. \tag{2.6}$$

Clearly, this partition function describes a model which is a $\mathbb{Z}_l$ orbifold of the original model (2.5); i.e. a boson on a circle of radius $\sqrt{2k}/l = \sqrt{2l'/l}$. Since $\psi_n = \psi_{-n}$, we have $Z_l = Z_{l'}$. Of course, this is a special case of T-duality.



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SU$(2)_k$ The irreducible representations are labeled by $j = 0, 1/2, \ldots, k/2$. The lowest $L_0$ eigenvalue in the j-th representation is $\Delta_j = \frac{j(j+1)}{k+2}$. The characters $\chi_j(q)$ transform under the non-trivial modular transformation $S$ as

$$\chi_j(q') = \sum_{j'} S_j^{\ j'} \chi_{j'}(q) , \qquad S_j^{\ j'} = \sqrt{\frac{2}{k+2}} \sin \frac{\pi(2j+1)(2j'+1)}{k+2} . \tag{2.7}$$

The fusion rules are

$$N_{jj'}^{\ \ j''} = \begin{cases} 1, & |j-j'| \le j'' \le \min\{j+j', k-j-j'\} \\ & \& \ j+j'+j'' \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

**Parafermions** $\mathcal{A}^{PF(k)} = \mathrm{SU}(2)_k / \mathrm{U}(1)_k$ The chiral algebra $\mathcal{A}^{PF(k)}$ of this theory has a set of irreducible representations $\mathcal{H}_{(j,n)}$ described by pairs $(j,n)$ where $j \in \frac{1}{2}\mathbb{Z}$, $0 \le j \le k/2$, and $n$ is an integer defined modulo $2k$. The pairs are subject to a constraint $2j + n = 0 \bmod 2$, and an equivalence relation $(j,n) \sim (\frac{k}{2} - j, k+n)$. We will denote the set of distinct irreducible representations by $PF(k) = \{(j,n)\}$. The character of the representation $(j,n)$, denoted $\chi_{jn}(q)$, is determined implicitly by the decomposition

$$\chi_j^{\mathrm{SU}(2)}(\tau, z) = \sum_{n=-k}^{k+1} \chi_{j,n}^{(k)}(q) \frac{\Theta_{n,k}(\tau, 2z)}{\eta(\tau)} , \tag{2.9}$$

where $q = \exp[2\pi i \tau]$. Explicitly we have [21]

$$\begin{aligned} \chi_{(j,n)}^{(k)}(\tau) &:= \mathrm{Tr}_{\mathcal{H}_{(j,n)}} q^{L_0 - c/24} \\ &= \frac{1}{\eta(\tau)^2} \left( \sum_{(x,y)\in A(j,n,k)} + \sum_{(x,y)\in A\left(\frac{k}{2}-j,k+n,k\right)} \right) \mathrm{sign}(x) e^{2\pi i \tau [(k+2)x^2 - ky^2]} , \end{aligned} \tag{2.10}$$

where

$$A(j,n,k) = \left\{ (x,y) : -|x| \le y \le |x|, \quad (x,y) \in \mathbb{Z}^2 + \left( j + \frac{1}{2(k+2)}, \frac{n}{2k} \right) \right\} . \tag{2.11}$$

The characters $\chi_{(j,n)}^{(k)}(\tau)$ are defined to be zero if $2j + n = 1 \bmod 2$. We often abbreviate the notation to $\chi_{jn}$, and the characters satisfy

$$\chi_{j,n} = \chi_{j,-n} = \chi_{\frac{k}{2}-j,k-n} . \tag{2.12}$$

(Warning: The representations labeled by $(j,n)$ and by $(j,-n)$ are distinct, but $(j,n)$ and $(\frac{k}{2} - j, k+n)$ are the same.) The lowest $L_0$ eigenvalue in the $(j,n)$ representation is

$$\Delta_{jn} = \begin{cases} \frac{j(j+1)}{k+2} - \frac{n^2}{4k} , & -j \le \frac{n}{2} \le j , \\ \frac{j(j+1)}{k+2} - \frac{n^2}{4k} + \frac{n-2j}{2} , & j \le \frac{n}{2} \le k-j , \end{cases} \tag{2.13}$$

where we shifted the fundamental domain of $n$ to be $n = -2j, -2j+2, \ldots, 2k-2j-2$ which automatically implements the selection rule $n+2j \in 2\mathbb{Z}$. Using (2.4) and (2.7) the action of the modular group on the characters is

$$\chi_{jn}(q') = \sum_{(j',n')\in PF(k)} S^{PF}_{(j,n)}{}^{(j',n')} \chi_{j'n'}(q) \,, \tag{2.14}$$

where $q' = \exp[-2\pi i/\tau]$, and the PF S-matrix is

$$S^{PF}_{(j,n)}{}^{(j',n')} = \sqrt{\frac{2}{k}} e^{i\pi n n'/k} S_j{}^{j'} \,. \tag{2.15}$$

The fusion rules of the theory are

$$N_{(jn),(j'n')}{}^{(j''n'')} := N_{jj'}{}^{j''} \delta_{n+n'-n''} + N_{jj'}{}^{\frac{k}{2}-j''} \delta_{n+n'-n''-k} \,, \tag{2.16}$$

where $\delta$ is a periodic delta function modulo $2k$.

When combining left and right-movers, the simplest modular invariant partition function of the parafermion theory is obtained by summing over all distinct representations

$$Z = \sum_{(j,n)\in PF(k)} |\chi_{jn}|^2 = \sum_j \sum_{n=1}^{2k} \frac{1}{2} |\chi_{jn}|^2 \,. \tag{2.17}$$

In the semiclassical limit as $k$ goes to infinity with fixed $n$ and $j$ only the primary states of the parafermion algebra survive. Also, from (2.13) it is clear that the spectrum includes states with every value of $j = 0, 1/2, 1, \ldots$ and for each $j$ the states are labeled by the integer $n$ satisfying $-j \le n/2 \le j$ and $2j+n \in 2\mathbb{Z}$. Their dimension is $\Delta_{jn} \approx \frac{1}{k}[j(j+1) - \frac{n^2}{4}]$.

### 2.2 Sigma model description

One of the main themes of this paper is that there are very natural and intuitive geometrical descriptions of the intricate algebraic formulae of RCFT. Accordingly, let us give the sigma model description of the results of the previous section.

We begin with the SU(2) level $k$ WZW model. We can write the metric on $S^3$ in the following ways, all of which will be useful.

$$\begin{aligned}
ds^2 &= d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\widetilde{\phi}^2 \,, \\
ds^2 &= \frac{1}{4}\left[(d\chi + \cos\widetilde{\theta} d\varphi)^2 + d\widetilde{\theta}^2 + \sin^2\widetilde{\theta} d\varphi^2\right] , \\
ds^2 &= d\psi^2 + \sin^2\psi ds^2_{S^2} \,, \\
\chi &= \widetilde{\phi} + \phi \,, \qquad \varphi = \widetilde{\phi} - \phi \,, \qquad \widetilde{\theta} = 2\theta \,, \\
g &= e^{i\chi\frac{\sigma^3}{2}} e^{i\widetilde{\theta}\frac{\sigma^1}{2}} e^{i\varphi\frac{\sigma^3}{2}} = e^{i2\psi\frac{\vec{n}.\vec{\sigma}}{2}} \,,
\end{aligned} \tag{2.18}$$

where the last line denotes the SU(2) group element in terms of the Euler angles. The ranges are $0 \le \widetilde{\theta} \le \pi$, $0 \le \varphi, \phi, \widetilde{\phi} \le 2\pi$, $0 \le \chi \le 4\pi$ while $0 \le \psi \le \pi$. In the first parametrization the sphere is presented as $|z_1|^2 + |z_2|^2 = 1$ with $z_1 = e^{i\phi}\sin\theta$; $z_2 = e^{i\widetilde{\phi}}\cos\theta$. Notice also that the angle $\psi$ is related to a rotation by angle $2\psi$ and $\vec{n}$ is a unit vector (a point on $S^2$).

The lagrangian for the SU(2) model can be written as

$$\begin{aligned} S &= k\int \partial\theta\overline{\partial}\theta + \sin^2\theta\partial\phi\overline{\partial}\phi + \cos^2\theta\partial\widetilde{\phi}\overline{\partial}\widetilde{\phi} + \sin^2\theta(\partial\phi\overline{\partial}\widetilde{\phi} - \overline{\partial}\phi\partial\widetilde{\phi}) \\ &= k\int \left[\partial\theta\overline{\partial}\theta + \tan^2\theta\partial\phi\overline{\partial}\phi + \cos^2\theta(\partial\widetilde{\phi} + \tan^2\theta\partial\phi)(\overline{\partial}\widetilde{\phi} - \tan^2\theta\overline{\partial}\phi)\right]. \end{aligned} \tag{2.19}$$

In the second line we essentially completed a square. Geometrically, the GKO coset amounts to gauging a U(1) symmetry of the model corresponding to shifting $\widetilde{\phi}$. This is achieved by adding a gauge field to the last term in the second line of (2.19) so that we get

$$\begin{aligned} S = k\int \Big[\partial\theta\overline{\partial}\theta + \tan^2\theta\partial\phi\overline{\partial}\phi + \cos^2\theta(\partial\widetilde{\phi} + \tan^2\theta\partial\phi + A_z) \times \\ \times (\overline{\partial}\widetilde{\phi} - \tan^2\theta\overline{\partial}\phi + A_{\overline{z}})\Big]. \end{aligned} \tag{2.20}$$

Integrating out $A$ removes completely the last term and produces a dilaton proportional to $\log\cos\theta$.

The net result is that the level $k$ parafermion theory is described by a sigma model with metric and string coupling

$$\begin{aligned} (ds)^2 &= k\frac{1}{1-\rho^2}(d\rho^2 + \rho^2 d\phi^2), \\ g_s(\rho) &\equiv e^{\Phi} = g_s(0)(1-\rho^2)^{-1/2}. \end{aligned} \tag{2.21}$$

The relation between the coordinates in the metric (2.19) and the metric (2.21) is $\rho = \sin\theta$. The topology of the target space is a disk. Geometrically it has finite radial geodesic distance, but infinite circumference. There is a curvature singularity at $\rho = 1$. From (2.21) it would appear that the sigma model has a U(1) symmetry of shifts of $\phi$. In fact the symmetry is broken to $\mathbb{Z}_k$. More precisely, we can check from (2.20) that the current corresponding to shifts in $\phi$ has a divergence proportional to $\partial_\alpha j^\alpha \sim kF_{z\overline{z}}$ where $F$ is the field strength of the gauge field. If we integrate this we see that angular momentum in the $\phi$ direction is violated by $k$ units. This is the familiar statement that if we gauge the vector U(1), then the axial U(1) is anomalous and vice versa. A $\mathbb{Z}_k$ subgroup of U(1) is non anomalous and is a good symmetry of the theory.

We can also relate this sigma model description to the usual algebraic description of the coset. First notice that $J_L^3$ and $J_R^3$ correspond to translations in the angles $\chi, -\varphi$, respectively (see (2.18)). In the parafermion theory we gauge $J_L^3 - J_R^3$, which corresponds to translations in $\widetilde{\phi}$. So starting from an SU(2) state $|\Psi\rangle$, we impose the constraints $J^3_{L,n}|\Psi\rangle = J^3_{R,n}|\Psi\rangle = 0$, for $n > 0$ and *also* $(J^3_{L,0} - J^3_{R,0})|\Psi\rangle = 0$. This is an explicit way to parametrize the states in the coset.

For large $k$ we can analyze the spectrum of low dimension operators by considering the "space-time" lagrangian

$$\sqrt{g}e^{-2\Phi}\frac{1}{2}g^{ab}\partial_a\Psi\partial_b\Psi \tag{2.22}$$

which depends on the field $\Psi$, and the metric and dilaton (2.21). $\Psi$ is the wave function of a massless field that has no indices in the directions of the parafermion space. The small fluctuations of $\Psi$ are controlled by the eigenvalue equation

$$\left[-\frac{1}{2}\nabla^2 + \nabla\Phi\nabla - 2\Delta\right]\Psi = 0 \tag{2.23}$$

(the factor of 2 in front of $\Delta$ is because the eigenvalue is $\Delta+\overline{\Delta} = 2\Delta$). We parametrize

$$\Delta_{jn} = \frac{j(j+1)}{k} - \frac{n^2}{4k} \tag{2.24}$$

and define

$$\Psi = z^{\frac{|n|}{2}}e^{in\phi}F(z)\,, \qquad z = \rho^2\,, \qquad n \in \mathbb{Z}\,. \tag{2.25}$$

Then $F$ satisfies the hypergeometric equation

$$z(1-z)F'' + [\gamma - (\alpha+\beta+1)z]F' - \alpha\beta F = 0 \tag{2.26}$$

with

$$\begin{aligned}
\alpha &= \frac{1}{2}\left(|n| + 1 + \sqrt{n^2+1+4k\Delta}\right) = \frac{|n|}{2} + j + 1\,,\\
\beta &= \frac{1}{2}\left(|n| + 1 - \sqrt{n^2+1+4k\Delta}\right) = \frac{|n|}{2} - j\,,\\
\gamma &= |n| + 1\,.
\end{aligned} \tag{2.27}$$

The boundary conditions on $F$ are as follows. Near $z = 0$ the space is smooth and from the form of $\Psi$ (2.25) we conclude that $F$ is analytic around that point. Therefore, $F$ is the hypergeometric function. The normalization condition on $\Psi$ is that $\int\sqrt{g}e^{-2\Phi}|\Psi|^2 \sim \int dz z^{|n|}F^2$ converges, and therefore we impose that $F(z=1)$ is finite. For $\gamma > \alpha+\beta$ $F(z=1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$. In our case $\gamma = \alpha+\beta = |n|+1$ and we have to consider the limit: $F(z=1) = \lim_{\epsilon\to 0}\frac{\Gamma(|n|+1)\Gamma(\epsilon)}{\Gamma(\frac{|n|}{2}-j+\epsilon)\Gamma(\frac{|n|}{2}+j+1)}$. It is finite only when

$$j = \frac{|n|}{2}, \frac{|n|}{2}+1, \ldots\,, \qquad n \in \mathbb{Z} \tag{2.28}$$

or equivalently $j = 0, 1/2, 1, \ldots$ and $-j \leq n/2 \leq j$. In this case $F$ is a polynomial in $z$ of degree $j - \frac{|n|}{2}$. See appendix D for further discussion of the eigenfunctions.

### 2.3 Orbifolds of the parafermion theory and T-duality

The parafermion theory has a $\mathbb{Z}_k$ global symmetry under which the fields $\psi_{j,n}$ generating the representation $(j,n)$ transform as

$$g : \psi_{j,n} \longrightarrow \omega^n \psi_{j,n}\,, \qquad \omega = e^{i2\pi/k}\,. \tag{2.29}$$

(Another $\mathbb{Z}_2$ symmetry is $\psi_{j,n} \to \psi_{j,-n}$.) Therefore, we can orbifold by any discrete subgroup of $\mathbb{Z}_k$. These are the $\mathbb{Z}_l$ subgroups generated by $g^{k/l}$ for $l$ that divides $k$. Taking a symmetric orbifold by $\mathbb{Z}_l$ of (2.17) leads to the partition function

$$Z = \frac{1}{2} \sum_{\substack{jn\overline{n} \\ n-\overline{n}=0 \mod 2l \\ n+\overline{n}=0 \mod 2l'}} \chi_{jn}\overline{\chi}_{j\overline{n}}\,, \tag{2.30}$$

where $l' = k/l$. One derives this by imposing the projection in the untwisted sector and then imposing modular invariance. Using the fact that $\chi_{j,n} = \chi_{j,-n}$ we see that the partition function of the orbifold of the theory by $\mathbb{Z}_l$ is the same as that of the orbifold by $\mathbb{Z}_{l'}$, suggesting the models are equivalent. Note, in particular, that the level $k$ parafermion theory would be equivalent to its $\mathbb{Z}_k$ orbifold. We will now argue that the models are equivalent.

The equivalence of the parafermion theory to its $\mathbb{Z}_k$ orbifold can in fact be seen at the sigma model level. Consider the the non-linear sigma model description of the theory based on the metric and dilaton (2.21). A simple way to see this is to perform T-duality on the U(1) isometry of the metric.[1]

The resulting sigma model is based on the metric and string coupling

$$\begin{aligned}
(ds')^2 &= \frac{k}{1-\rho'^2}(d\rho'^2 + \rho'^2 d\phi'^2)\,, \\
g'_s &= \frac{g_s(0)}{\sqrt{k}}(1-\rho'^2)^{-1/2}\,, \\
\rho' &= (1-\rho^2)^{1/2}\,, \\
\phi' &\sim \phi' + \frac{2\pi}{k}.
\end{aligned} \tag{2.31}$$

In other words, the original model with string coupling $g_s(0)$ is T-dual to its orbifold with string coupling $g_s(0)/\sqrt{k}$. Note that the transformation on $\rho$ exchanges the boundary of the disk and its center. A similar T-duality transformation establishes the more general relation mentioned above between a $\mathbb{Z}_l$ orbifold of the theory and its $\mathbb{Z}_{l'}$ orbifold.

[1] Strictly speaking this is not correct since the system does not really have a U(1) symmetry, for reasons we explained above. A better way to derive this equivalence is by performing the usual steps that implement a T-duality (as in [22]) in the gauged WZW model (2.20), before integrating over $A$. This also proves it to all string loops. The net result is the same as performing the naive T-duality.

This T-duality also sheds light on the lack of global U(1) symmetry of the theory. The metric and dilaton of the theory (2.21) are invariant under U(1). How is it that the theory is not invariant under U(1)? The point is that close to $\rho = 1$ the sigma model description is singular because the metric and the string coupling diverge there. The T-duality transformation allows us to explore this strongly coupled region. In the T-dual variables (2.31) this region is mapped to $\rho' = 0$, where the only singularity is an order $k$ orbifold singularity of the metric. The putative U(1) symmetry associated with momentum around the disk is mapped to winding symmetry around the origin. The orbifold singularity makes it clear that winding is conserved modulo $k$. We conclude that the T-dual variables exhibit the breaking of the original U(1) symmetry to $\mathbb{Z}_k$ near the boundary of the disk where the original variables are strongly coupled.

## 3. D-branes in the parafermion theory

### 3.1 $\mathrm{U}(1)_k$ theory

This theory is a useful warm up example. The D-branes in this theory are well-known (see for example, [6]). We find here two kinds of boundary states A-states and B-states which are annihilated by $J(\sigma) \pm \overline{J}(\sigma)$ ($\sigma$ is a parameter around the boundary). The terminology of A-branes and B-branes is as in $N = 2$ superconformal field theories [18] (see below). The Ishibashi A states which preserve the whole chiral algebra are $|A, r, r\rangle\rangle$ with $r = -k+1, -k+2, \ldots, k$. The two integers denote the momentum of the left moving and the right moving U(1) algebras. For clarity, we give the explicit expressions for the A,B-type Ishibashi states. The A-Ishibashi states are

$$|Ar, r\rangle\rangle = \exp\left[+\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\widetilde{\alpha}_{-n}\right]\sum_{\ell\in\mathbb{Z}}\left|\frac{r+2k\ell}{\sqrt{2k}}, \frac{r+2k\ell}{\sqrt{2k}}\right\rangle \tag{3.1}$$

while the B-Ishibashi states are

$$|Br, -r\rangle\rangle = \exp\left[-\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}\widetilde{\alpha}_{-n}\right]\sum_{\ell\in\mathbb{Z}}\left|\frac{r+2k\ell}{\sqrt{2k}}, -\frac{r+2k\ell}{\sqrt{2k}}\right\rangle . \tag{3.2}$$

The condition that the momentum states in the right-hand side of (3.2) exist in the closed string spectrum is $r/k \in \mathbb{Z}$; i.e. $r = 0, k$. It should be stressed that $|A, 0, 0\rangle \neq |B, 0, 0\rangle$.

Using the A-Ishibashi states we can form $2k$ Cardy states

$$|A, \widehat{n}\rangle_C = \frac{1}{(2k)^{1/4}}\sum_{n'=0}^{2k-1} e^{-i\pi\widehat{n}n'/k}|A, n', n'\rangle\rangle . \tag{3.3}$$

They are interpreted geometrically as D0-branes at $2k$ special points on the circle.

Linear combinations of the two B states (3.2) are the two B Cardy states

$$|B,\eta=\pm 1\rangle_C = \left(\frac{k}{2}\right)^{1/4}\left[|B,0,0\rangle\rangle + \eta|B,k,-k\rangle\rangle\right] \tag{3.4}$$

which we interpret as D1-branes with special values of the Wilson line parametrized by $\eta$.

The B-branes can also be found by considering the T-dual theory which is its $\mathbb{Z}_k$ orbifold. The A-branes in this theory are obtained from (3.3) by summing over the images. We find two states corresponding to D0-branes at the two special points. After T-duality to the original model they are mapped to two B-branes which can be interpreted as D1-branes with two special values of the Wilson lines.

This theory also has other D-branes which correspond to D0-branes at arbitrary positions around the circle and D1-branes with arbitrary values of the Wilson line. These more generic boundary states are perfectly consistent but the algebraic considerations based on the $\mathrm{U}(1)_k$ chiral algebra do not reveal them because they are not invariant under such a large chiral algebra but only under a smaller subalgebra. Specifically, they are invariant under a chiral algebra which is generated by the current $J(\sigma)$ and its derivatives but without the exponential of the compact boson. A signal of these more generic D-branes is found by examining the spectrum of open strings on the A and B branes mentioned above. They all have a massless ($\Delta=1$) open string propagating on them. It is easy to check that this open string leads to a modulus and therefore these D-branes belong to a moduli space of such D-branes.

The special case of $k=1$ is particularly interesting. Here $\mathrm{U}(1)_1 = \mathrm{SU}(2)_1$. In addition to the two circles of D-branes we discussed above (D0 at an arbitrary position and D1 with an arbitrary Wilson line) there are more D-branes. Here all the D-branes are on an $S^3$. A signal of this larger moduli space is again found by noticing the three massless moduli on the D-branes which are in an SU(2) triplet.

### 3.2 A-branes in the parafermion theory

As in the previous example, the A-branes are invariant under the full parafermion symmetry and are found by following the standard procedure. The parafermion Ishibashi states $|A,j,n\rangle\rangle$ are constructed from the same representation $(j,n)\in PF(k)$ on the left and right. These Ishibashi states are in one to one correspondence with the primaries of the chiral algebra and they are related to the Cardy states

$$|A,\widehat{j},\widehat{n}\rangle_C = \sum_{(j,n)\in PF(k)} \frac{S^{PF\ jn}_{\widehat{j}\widehat{n}}}{\sqrt{S^{PF\ jn}_{00}}}|A,j,n\rangle\rangle\,. \tag{3.5}$$

We can calculate the open string spectrum in the standard way using the parafermion characters $\chi_{jn}$, their modular transformation properties and the Verlinde formula.

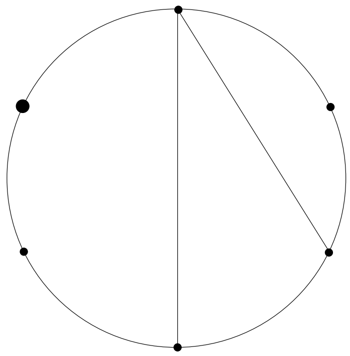

**Figure 1:** These are various A-branes in the $k = 6$ case. We have a disk with $k$ special points along the boundary. We can see a D0-brane at one of the special points and some D1-branes stretched between different special points.

We find

$$ {}_C\langle A, \widehat{j}, \widehat{n} | q'^{L_0 + \overline{L}_0 - \frac{c}{12}}_c | A, \widehat{j}', \widehat{n}' \rangle_C = \sum_{(j,n) \in PF(k)} N^{jn}_{\widehat{j}, -\widehat{n}, \widehat{j}' \widehat{n}'} \chi(q_o)_{jn} \,, \tag{3.6} $$

where $q_o = \exp[-2\pi i/\tau]$ is the open string modular parameter and $q_c \equiv q'^2_c \equiv e^{2\pi i \tau}$ is the closed string parameter.

These D-branes can be interpreted as follows. The $k$ states with $\widehat{j} = 0$ are D0-branes at one of the $k$ special points around the disk. The $k[\frac{k-1}{2}]$ states with $\widehat{j} = 1/2, 1, \ldots, \frac{1}{2}[\frac{k-1}{2}]$ are unoriented D1-branes stretched between two of the special points around the disk separated by $2\widehat{j}$ segments. Finally, for $k$ even we also have $k/2$ branes with $\widehat{j} = k/4$ stretched between two antipodal special points (see figure 1).

This interpretation is supported by the following observations:

(1) We explicitly compute the shape of the branes in appendix D by scattering closed strings.

(2) The $\mathbb{Z}_k$ global symmetry multiplies the Ishibashi states by a phase and shifts the $\widehat{n}$ label of the Cardy states, $\widehat{n} \to \widehat{n} + 2$. Since this symmetry rotates the disk, it moves a D1-brane connecting two points separated by $2\widehat{j}$ segments to another D1-brane separating $2\widehat{j}$ segments.

(3) For large $\widehat{j}$ and $k$ the dynamics of D1-branes can by analyzed by their DBI action. The effective metric for a D1-brane is

$$ \frac{(ds)^2}{g_s^2} = k(d\rho^2 + \rho^2 d\phi^2) \,. \tag{3.7} $$

Since it is flat, the D1-branes are straight lines.

(4) The masses of the D-branes are found from their overlap with $|A, j = 0, n = 0\rangle\rangle$, since the graviton vertex operator in the non-compact spacetime is proportional

to the unit operator in the internal theory. Thus the masses are proportional to

$$M_{\widehat{j}} := \frac{1}{g_s(0)\sqrt{k(k+2)}} \sin \frac{\pi(2\widehat{j}+1)}{k+2} \,. \tag{3.8}$$

This expression is consistent with the geometric interpretation suggested above. For large $\widehat{j}$ and $k$ the mass of the D-brane is proportional to $\sin \frac{2\pi\widehat{j}}{k}$, which is the same as the length of the D-brane in the flat metric on the disk (3.7). (The mass is a regularized dimension of a Hilbert space, in accord with the general result [23].)

(5) We can also see that if we take a conjugacy class in SU(2) that is invariant under the U(1) symmetry that we are gauging, then we can define a brane in the coset theory. This procedure gives straight lines on the disk. To see this more precisely we think of $S^3$ as $|z_1|^2 + |z_2|^2 = 1$ the U(1) we gauge is the phase of $z_2$. The disk is parametrized by $z_1$. A conjugacy class is given by intersecting the $S^3$ by a hyperplane in $R^4$. We get the straight lines if we intersect by hyperplanes of the form $az_1 = b$ where $a, b$ are constants. These conjugacy classes are invariant under phase rotations of $z_2$ and they project to straight lines on the disk.

### 3.3 Geodesics and open strings

As a nice check on the open string spectrum between branes given by (3.6), and as supporting evidence for the above geometrical interpretation of the states $|A, \widehat{j} = 0, \widehat{n}\rangle_C$ we will compute the action for a fundamental string in the metric (2.21) stretching from $(\rho = 1, \phi)$ to $(\rho = 1, \phi + \frac{\pi n}{k})$, $n \in 2\mathbb{Z}$. Accordingly, we compute the length of the geodesic stretching between two points on the boundary of the disk separated by an angle $\Delta\phi$. Such a geodesic emerges orthogonally from the boundary, reaches a minimal value of $z = \rho^2$, and returns orthogonally to the boundary.

The geodesic equation in the metric (2.21) implies

$$\frac{\dot{\rho}^2}{\rho^2} = \frac{\rho^2 - \rho_m^2}{\rho_m^2(1-\rho^2)} \,, \tag{3.9}$$

where $\dot{\rho} = d\rho/d\phi$ and $\rho_m$ is an integration constant which we took such that it is the minimum value of $\rho$ along the geodesic. From this we find that

$$\frac{\Delta\phi}{2} = \frac{\sqrt{z_m}}{2} \int_{z_m}^{1} dz \frac{\sqrt{1-z}}{z\sqrt{z-z_m}} = \frac{\pi}{2}(1-\sqrt{z_m}) \,. \tag{3.10}$$

Now we can compute the length of the geodesic. We get

$$\ell = \sqrt{k}\sqrt{1-z_m} \int_{z_m}^{1} dz \frac{1}{\sqrt{(z-z_m)(1-z)}} = \pi\sqrt{1-z_m} \,. \tag{3.11}$$

We now use the relation between the length and $L_0$ to find (We use units where $\alpha' = 1$)

$$L_0 = \left(\frac{\ell}{(2\pi)}\right)^2 = \frac{k}{4}\left(2\left(\frac{\Delta\phi}{\pi}\right) - \left(\frac{\Delta\phi}{\pi}\right)^2\right). \tag{3.12}$$

For an angle

$$\Delta\phi = \pi\frac{n}{k} \tag{3.13}$$

we get $L_0 = \frac{n(2k-n)}{4k}$ in perfect accord with the conformal dimension of the lightest open string, whose dimension is given by the dimension of the corresponding parafermion representation $\Delta(j = 0, n)$ in the open string channel. The conformal field theory expression is exact, and yet is a finite power series in $1/k$. Therefore, the semiclassical analysis should give the exact answer, as indeed it does.

### 3.4 B-branes

In this subsection we will exhibit new branes in the parafermion theory. We will find them using the same approach we took to find the B-branes in the $\mathrm{U}(1)_k$ theory above. We will use the fact that the theory is T-dual to its $\mathbb{Z}_k$ orbifold. A-branes in the orbifold theory are found from the A-branes in its covering theory using standard orbifold techniques. Then the T-duality maps these branes to B-branes in the original theory.

**Qualitative considerations** We look for the spectrum of A-branes in the $\mathbb{Z}_k$ orbifold of the parafermion theory with string coupling $g_s(0)$. We start with the branes described above and we add the images under $\mathbb{Z}_k$ so that the configuration is $\mathbb{Z}_k$ invariant. We find a single state for each value of $\widehat{j} = 0, 1/2, 1, \ldots, \frac{1}{2}[\frac{k-1}{2}]$ with mass $M_{\widehat{j}}$ as in (3.8) (we do not need to multiply the masses by $k$ for the $k$ images because we view the target space as the disk modded out by $\mathbb{Z}_k$ and do not compare to the mass of a D-brane in the theory on the covering space). When $k$ is even there are also two D1-branes of mass $\frac{1}{2}M_{\widehat{j}=k/4}$ which are stretched from the special point to the center of the disk mod $\mathbb{Z}_k$. In terms of the theory on the covering space they are obtained by taking $k/2$ images (hence the factor of $1/2$ in their mass). There are two different such D-branes because they are like fractional D-branes at fixed points of orbifolds.

This $\mathbb{Z}_k$ orbifold theory with string coupling $g_s(0)/\sqrt{k}$ is T-dual to the parafermion theory with string coupling $g_s(0)$. Therefore, each of these D-branes should have a counterpart in the parafermion theory. The transformation $\rho' = (1-\rho^2)^{1/2}$ (2.31) shows that these D-branes which are near the boundary of the disk in the orbifold are near the center in the parafermion theory. Therefore, these are new D-branes not included in the list of $k(k+1)/2$ D-branes mentioned above.

The A-brane with $\widehat{j} = 0$ in the orbifold theory is unstable. The open string stretched between it and its adjacent image includes a tachyon.[2] We will see that in more detail below when we analyze the spectrum of open strings in detail. The instability corresponds to motion of the D0-A-brane to the center of the orbifold. This brane in the orbifold theory maps to a D0 B-brane at the center of the disk. The instability corresponds to the possibility of moving the brane off the center. This lowers the energy since the string coupling has a minimum at the center of the disk. Since all other B-branes are also located at the center we expect that they should also be unstable under displacements off the center of the disk. In terms of the A-branes in the orbifold theory this instability is easy to understand. The A-D1-branes with $\widehat{j} = 1/2, 1, \ldots, \frac{1}{2}[\frac{k-1}{2}]$ form, in the covering space, closed loops. They can clearly decay to smaller closed loops which are closer to the origin.[3]

The two states with $\widehat{j} = k/4$ which exist for $k$ even correspond in the covering space to $k/2$ D1-branes stretched between antipodal points on the disk. They cannot decay by shrinking and therefore they are stable.

Remembering the value of $g_s(0)$ we conclude that the parafermion theory also has the following D-brane states near the center of the disk. There is a D0-brane with $\widehat{j} = 0$ and mass $\sqrt{k} M_{\widehat{j}=0}$. There are D2-branes around the center for $\widehat{j} = 1/2, 1, \ldots, \frac{1}{2}[\frac{k-1}{2}]$ with masses $\sqrt{k} M_{\widehat{j}}$. The fact that these are D2-branes can be seen either by following the T-duality transformation or by thinking, for large $k$, about the way the large number of images go through the disk mod $\mathbb{Z}_k$ (see figure 2). All these D-branes are unstable. Finally, there are two more stable D2-branes with masses $\frac{1}{2}\sqrt{k} M_{k/4}$. As is clear from the configuration of the D1-branes in the orbifold theory, these two D2-branes cover the whole disk.

**Boundary state formulae for the B-branes** Let us now use the above qualitative picture to give a concrete formula for the new D2-branes centered on the disk. We must first apply the orbifold projection to (3.5) and then implement a T-duality. The orbifold projection keeps only the $n = 0$ Ishibashi states (the $n = k$ states can be expressed as $n = 0$ using the $(j, n) \sim (\frac{k}{2} - j, n + k)$ identification).

Now let us implement the $T$-duality. Roughly speaking, we simply want to change the momentum of the rightmovers relative to that of the leftmovers. More precisely, we proceed as follows.

The coset construction implies an equality of CFT state-spaces

$$\mathcal{H}_j^{\mathrm{SU}(2)} = \oplus_{r=0}^{2k-1} \mathcal{H}_{(j,r)}^{PF} \otimes \mathcal{H}_r^{\mathrm{U}(1)_k} \tag{3.14}$$

[2]Since we are considering a bosonic system there are many other tachyons. In particular, the open strings which start and end on the same D-brane always include tachyons corresponding to functions on the space. This is the standard instability of the bosonic string and does not signal the instability of this brane configuration. We will ignore it in this discussion.

[3]Some aspects of brane decay appeared while this paper was being written [24]. See [25] for similar discussions in Virasoro minimal models.



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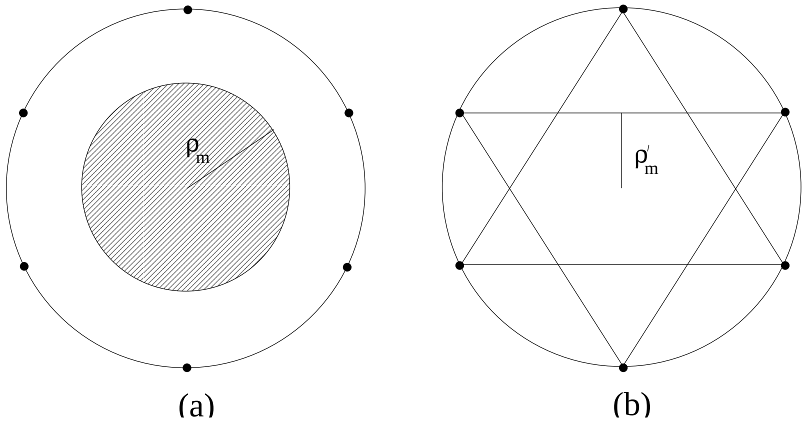


**Figure 2:** In figure (a) we see a B-branes at the center of the disk. It can be viewed as arising from the A-branes of the $\mathbb{Z}_k$ quotient of the parafermion theory by T-duality. The covering space of this theory is depicted in figure (b) along with a $\mathbb{Z}_k$ invariant configuration of A-branes of this theory. The T-dual of this configuration is the D2-brane in figure (a). The distances $\rho_m$ and $\rho'_m$ are related through $\rho_m^2 + \rho_m'^2 = 1$.

and there is accordingly a decomposition of A-type Ishibashi states: $|j\rangle\rangle = \sum_n |j,r,r\rangle\rangle|r,r\rangle\rangle$. We now note that the operator $e^{i\pi J_0^1}$ is block-diagonal in the decomposition (3.14) since

$$e^{i\pi J_0^1} J_0^3 e^{-i\pi J_0^1} = -J_0^3 \,. \tag{3.15}$$

Moreover, this operator may be defined to act on states in the $\mathrm{U}(1)_k$ theory by $\alpha_n \to -\alpha_n$ and $|p\rangle \to +|-p\rangle$ (the choice of phase on the vacuum is a convention). This defines, implicitly, an action

$$e^{i\pi J_0^1} : \mathcal{H}^{PF}_{(j,r)} \longrightarrow \mathcal{H}^{PF}_{(j,-r)} \,. \tag{3.16}$$

We now define the B-type parafermion Ishibashi states by

$$(1 \otimes e^{i\pi \widetilde{J}_0^1})|j\rangle\rangle^{\mathrm{SU}(2)} := \sum_{r=0}^{2k-1} |Bj,r,-r\rangle\rangle^{PF} \otimes |B,r,-r\rangle\rangle^{\mathrm{U}(1)_k} \,. \tag{3.17}$$

It is an Ishibashi state that preserves the chiral algebra up to a T-duality automorphism, the automorphism that changes $n \to -n$.

We may now write the boundary Cardy state for the B-branes:

$$|B,\widehat{j}\rangle_C = \sqrt{k} \sum_{j\in\mathbb{Z}} \frac{S^{PF\ j0}_{\widehat{j}0}}{\sqrt{S^{PF\ j0}_{\widehat{0}0}}} |B,j,0\rangle\rangle = (2k)^{1/4} \sum_{j\in\mathbb{Z}} \frac{S_{\widehat{j}}^{\ j}}{\sqrt{S_0^{\ j}}} |B,j,0\rangle\rangle \,. \tag{3.18}$$

There is a very interesting subtlety which arises when $k$ is even. Below we will analyze the open string spectrum on the D-branes. We will find that on $|B,\widehat{j}=k/4\rangle_C$ as defined in (3.18), the identity operator appears twice. The reason for this is that this boundary state is "reducible," that is, it can be written as a sum of two more fundamental D-branes. Indeed, for even $k$ we can form another Ishibashi state:

$$\left|B, \frac{k}{4}, \frac{k}{2}, -\frac{k}{2}\right\rangle\!\!\Big\rangle\,, \tag{3.19}$$

where we have indicated the left and right moving $n$ quantum numbers for greater clarity. This B-state is in the closed string sector because after using the spectral flow identification on the left part it is clear that its quantum numbers are such that $n = \overline{n}$. Accordingly, we construct new irreducible D2 B-branes as

$$\left|B, \frac{k}{4}, \eta\right\rangle_C = \frac{1}{2}(2k)^{1/4}\sum_{j\in\mathbb{Z}}\frac{S_{\widehat{j}}^{\ j}}{\sqrt{S_0^{\ j}}}|B, j, 0\rangle\rangle + \eta\frac{1}{2[k(k+2)]^{1/4}}\left|B\frac{k}{4}, \frac{k}{2}, -\frac{k}{2}\right\rangle\!\!\Big\rangle\,, \tag{3.20}$$

where $\eta = \pm 1$ and we fixed the coefficient of the last term by demanding that we get a sensible open string spectrum.

We mentioned above that the global $\mathbb{Z}_k$ symmetry multiplies the Ishibashi state labeled by $n$ by a phase $e^{2\pi i n/k}$, and therefore the A Cardy states are shifted $|A, \widehat{j}, \widehat{n}\rangle_C \to |A, \widehat{j}, \widehat{n}+2\rangle_C$. The B Cardy states with $\widehat{j} < k/4$ are clearly invariant. But since $|B, k/4, k/2, -k/2\rangle\rangle \to -|B, k/4, k/2, -k/2\rangle\rangle$, the two states with $j = k/4$ transform to one another $|B, k/4, \eta\rangle \to |B, k/4, -\eta\rangle$.

**Properties of the B boundary states** We must now check that the states (3.18) and (3.20) satisfy the Cardy condition and therefore have good gluing properties. In the process we will compute the open string spectra on these D-branes and confirm the geometrical picture suggested above.

The overlap between two states of type (3.18) is easily computed because the overlap between two B Ishibashi states is the same as between two A states so we get

$$\begin{aligned}
{}_C\langle B\widehat{j}|q_c^{L_0-\frac{c}{24}}|B\widehat{j}'\rangle_C &= (2k)^{1/2}\sum_{j\in\mathbb{Z}}\frac{S_{\widehat{j}}^{\ j}S_{\widehat{j}'}^{\ j}}{S_0^{\ j}}\chi_{j,0}(q_c) \qquad (3.21)\\
&= \sum_{(j'',n'')\in PF(k)}\sum_{j\in\mathbb{Z}}\frac{S_{\widehat{j}}^{\ j}S_{\widehat{j}'}^{\ j}S_j^{\ j''}}{S_0^{\ j}}\chi_{j'',n''}(q_o) = \sum_{j''=0}^{k/2}\sum_{n''=0}^{2k-1}N_{\widehat{j},\widehat{j}'}^{j''}\chi_{j'',n''}(q_o)\,.
\end{aligned}$$

(The fact that the coefficients are integers is a check on the normalization in (3.18).) In deriving this formula we used the fact that

$$\sum_{j\in\mathbb{Z}}\frac{S_{\widehat{j}}^{\ j}S_{\widehat{j}'}^{\ j}S_j^{\ j''}}{S_0^{\ j}} = \sum_{\text{all } j}\frac{S_{\widehat{j}}^{\ j}S_{\widehat{j}'}^{\ j}S_j^{\ j''}}{S_0^{j}}\frac{(1+(-1)^{2j})}{2} = \frac{1}{2}\left(N_{\widehat{j},\widehat{j}'}^{j''} + N_{\widehat{j},\widehat{j}'}^{\frac{k}{2}-j''}\right)\,, \tag{3.22}$$

where we used $S_j^{\ \frac{k}{2}-j'} = S_j^{\ j'}(-1)^{2j}$. In the final step in (3.21) we also used that the sum over $j'', n''$ includes all parafermionic states twice.

We must now consider the overlap between the new B states and the previously considered $|A\widehat{j}\widehat{n}\rangle_C$ states corresponding to D0 and D1-branes. This is an involved computation whose details are given in appendix B. The result is

$$ {}_C\langle B,\widehat{j}'|q_c^{L_0-\frac{c}{24}}|A,\widehat{j}\widehat{n}\rangle_C = \sum_{j''} N^{j''}_{\widehat{j}',\widehat{j}}\widetilde{\chi}_{j''}(q_o)\,, \tag{3.23} $$

where $\widetilde{\chi}_j$ is a certain modular function defined in appendix B.

As we have mentioned, when $k$ is even there are two B-branes with $\widehat{j} = k/4$. Their overlaps are easily computed to be

$$ {}_C\left\langle B,\frac{k}{4}\eta\right| q_c^{L_0-\frac{c}{24}} \left|B,\frac{k}{4}\eta'\right\rangle_C = \frac{1}{4}\sum_{\substack{jn\\ j\in\mathbb{Z}}}\chi_{jn}(1+\eta\eta'(-1)^{j-\frac{n}{2}}) = \sum_{\substack{(j,n)\in PF(k)\\ j\in\mathbb{Z}\\ j-\frac{n}{2}\in 2\mathbb{Z}+\frac{1-\eta\eta'}{2}}} \chi_{jn}\,, \tag{3.24} $$

where in the first sum we sum over all $j,n$ ($j\in\mathbb{Z}$) and in the second we restrict the sum over distinct states (i.e. we do not sum over spectral flow images, as implied by $(j,n)\in PF(k)$). Similarly we can compute the open string spectrum involving the other B-branes and we find, for $\widehat{j}\neq k/4$

$$ \langle B,\widehat{j}|q_c^{L_0-\frac{c}{24}} \left|B,\frac{k}{4}\eta\right\rangle_C = \frac{1}{2}\sum_{jn} N^{j}_{\widehat{j},\frac{k}{4}}\chi_{jn} \tag{3.25} $$

which is a sensible spectrum. Each state appears twice in the sum, and therefore all states appear with integer coefficients (actually the coefficients are one).

The overlaps of the irreducible B-branes with $\widehat{j} = k/4$ with the A-branes can be computed using techniques described in appendix B, but we have not carried out the details.

The above overlaps show that the Cardy conditions are satisfied, as expected from the orbifold construction. The fact that we obtained the branes from an orbifold construction ensures that they obey all open strings consistency conditions, so that they are good boundary states.

Finally, we consider the spacetime masses of these branes. Notice that the mass of the B states with Cardy spin $\widehat{j}$ is the same as $\sqrt{k}$ times the mass of the A state with Cardy spin $\widehat{j}$, again in perfect accord with the orbifold construction.

In appendix C we work out explicitly some examples of this construction for low values of $k$.

One can also consider branes in various orbifolds of the parafermion theories. It is interesting to note that in the $\mathbb{Z}_l$ orbifold of the parafermion theory the first $l-1$ B-branes become stable. For example consider the B-brane with $\widehat{j}=0$. It is a D0-brane at the center of the disk. In the original parafermion theory this brane is unstable under small displacements from the center, since the center is where the string coupling is smallest. If we orbifold by $\mathbb{Z}_2$ this brane becomes stable since the projection removes the mode that would lead to the instability.

## 4. Other descriptions of the B-branes

### 4.1 Bound states of D0-branes

In the case of D-branes in the WZW model, for $k \gg 1$ we can interpret the spherical D2-branes for $\widehat{j} \ll k$ as states in the theory of $N = 2\widehat{j} + 1$ D0-branes. These D0-branes are described in terms of $N \times N$ matrices with non-commutative expectation values [26].

A similar description is possible in the parafermion theory. In fact this was done in [19] in a slightly different context. The potential for $N$ D0-branes at the center of the disk has the form (up to numerical constants)

$$V \sim \int \mathrm{Tr}\left[ -\frac{1}{k}(X_1^2 + X_2^2) - [X_1, X_2]^2 + \cdots \right], \tag{4.1}$$

where $X_1$ and $X_2$ are $N \times N$-dimensional matrices. The first term comes from expanding the dilaton potential $\sim \sqrt{1-\rho^2}$. They have a minus sign because the dilaton has a minimum at the origin. The last term is the usual commutator term that is present also in flat space. We neglected higher order terms. A solution to the equations of motion is

$$X_1 \sim \frac{1}{\sqrt{k}} J_1 \,, \qquad X_2 \sim \frac{1}{\sqrt{k}} J_2 \,, \tag{4.2}$$

where $J_1$ and $J_2$ are SU(2) matrices in an $N$-dimensional representation. The D2 disks we considered above correspond to taking irreducible representations of spin $\widehat{j}$ ($N = 2\widehat{j} + 1$). Notice that these are not stable minima as the potential (4.1) is unbounded below. We see that the description of these states is very similar to the description of the D2 states in SU(2). In this way of viewing them it is clear that the disks are flattened spheres and have two sides.

### 4.2 Effective action of D2-B-branes

In this subsection we will show that B-branes with $\widehat{j} \gg 1$ can be viewed as flattened disks with induced D0 charge. The disk has two sides and can be viewed as a flattened two-sphere.

We consider the classical equations for a disk D2-brane with an $F$-field on it. We imagine a fixed flux of $F$ on the disk

$$\frac{N}{k} = \frac{(2\widehat{j}+1)}{k} = 2\frac{1}{2\pi}\int_{D_2} \frac{F}{k} = \frac{1}{\pi}\int_0^{\rho_m} d\rho f(\rho) \,, \qquad \frac{2\pi F_{\rho\varphi}}{k} = f(\rho) \tag{4.3}$$

(we use units with $\alpha' = 1$). The factor of 2 in the third expression arises since our disk is made of two overlapping branes — a squashed $S^2$. Holding this fixed we want to minimize the DBI action. So we minimize

$$S - \lambda\left(\int F - N\right) = \frac{k}{2\pi}\int d\varphi \rho \sqrt{1-\rho^2}\sqrt{\frac{\rho^2}{(1-\rho^2)^4 + f^2}} - k\lambda\left(\int f - \frac{N}{k}\right), \tag{4.4}$$

where $\lambda$ is a constant Lagrange multiplier. The equation for $f$ that we get from this minimization procedure is

$$f = \frac{\rho}{(1-\rho^2)} \frac{\sqrt{1-\rho_m^2}}{\sqrt{\rho_m^2-\rho^2}} , \tag{4.5}$$

where $\rho_m$ is the maximum value of $\rho$. The Lagrange multiplier is a function of $\rho_m$ which is of no importance. We determine $\rho_m$ through the condition (4.3) and we get

$$\int_0^{\rho_m} d\rho f(\rho) = \theta_m , \qquad \tan\theta_m = \frac{\rho_m}{\sqrt{1-\rho_m^2}} . \tag{4.6}$$

So we see that $\theta_m$ is related to $\widehat{j}$ by

$$\frac{(2\widehat{j}+1)}{k} = \frac{\theta_m}{\pi} . \tag{4.7}$$

We see that as $\rho_m \to 1$ then $\widehat{j} \to k/4$ and $\rho_m \to 1$ so that the brane is covering the whole space. Note that from (4.5) this brane has zero $F$, but the integral of $F$ is still non zero! (There is an order of limits problem at the boundary).

We can calculate the open string metric and open string coupling using the transformation formulas of [27] (taking $B \to B + F$ and setting $B = 0$) we get

$$\begin{aligned} ds^2_{\text{open}} &= k\frac{d\rho^2 + \rho^2 d\varphi^2}{\rho_m^2 - \rho^2} = k\frac{d\rho'^2 + \rho'^2 d\varphi^2}{1-\rho'^2} , \\ e^D &\equiv G_s(\rho) = G_s(0)(1-\rho'^2)^{-1/2} , \\ \rho' &= \frac{\rho}{\rho_m} . \end{aligned} \tag{4.8}$$

We see that the open string metric and coupling on the B-brane are the same as the original closed string parameters. In particular, they are independent of $\rho_m$, which parametrizes the size of the brane.

We conclude that the B-brane ranges from the origin $\rho = 0$ to $\rho = \rho_m = \sin\theta_m = \sin(2\widehat{j}+1)\pi/k$. This conclusion can also be reached by considering the T-dual picture we originally used to derive the existence of these branes. In this picture the target space is a wedge and these branes are D1-branes (except for $\widehat{j} = 0$). In the covering space these D-branes are made out of $k$ D1-branes (except for $\widehat{j} = k/4$ when there are only $k/2$ D1-branes) stretched between points at an angle of $4\widehat{j}\pi/k$ (see figure 2). These D1-branes are in an annulus ranging from $\rho = \cos(2\widehat{j}\pi/k)$ to $\rho = 1$. Using the change of variables (2.31) $\rho \to \sqrt{1-\rho^2}$ in the T-duality, we learn that these D-branes are D2-branes ranging from $\rho = 0$ to $\rho = \sin(2\widehat{j}\pi/k)$ in the parafermion theory. This agrees with the expression for $\rho_m$ derived above (recall that this discussion makes sense only for large $\widehat{j}$ and $k$).

Using this metric and dilaton we should be able to find the spectrum of small fluctuations on the D2-branes as follows. We consider a field $\Psi$ moving under the influence of the lagrangian

$$\sqrt{G}e^{-D}G^{ab}\partial_a\Psi\partial_b\Psi \tag{4.9}$$

with the open string metric and dilaton given in (4.8). The small fluctuations of $\Psi$ are controlled by the eigenvalue equation

$$[-\nabla^2 + \nabla D \nabla - \Delta]\Psi = 0\,. \tag{4.10}$$

The relative factor of 4 between $\Delta$ here and in (2.23) is common in comparisons of open strings and closed strings. In terms of $z = \rho'^2$ we find that

$$\begin{aligned}
\Psi &= z^{|n|/4}e^{i\frac{n}{2}\phi}F(\alpha,\beta;\gamma;z)\,, \qquad n\in 2\mathbb{Z}\,,\\
\alpha &= \frac{1}{2}\left(\frac{|n|+1}{2} + \sqrt{\frac{n^2+1}{4}+k\Delta}\right) = \frac{|n|}{4}+\frac{j}{2}+\frac{1}{2}\,,\\
\beta &= \frac{1}{2}\left(\frac{|n|+1}{2} - \sqrt{\frac{n^2+1}{4}+k\Delta}\right) = \frac{|n|}{4}-\frac{j}{2}\,,\\
\gamma &= \frac{|n|}{2}+1
\end{aligned} \tag{4.11}$$

solves (4.10) where $F$ is a hypergeometric function and we parametrized

$$\Delta_{jn} = \frac{j(j+1)}{k} - \frac{n^2}{4k}\,. \tag{4.12}$$

In order to find the eigenvalues we need to impose a boundary condition. It is not clear which is the correct boundary condition. If we impose the boundary condition that $\Psi$ vanishes at $\rho' = 1$, then we get that $j = \frac{n}{2} + 2s$, $s \in \mathbb{Z}$. On the other hand if we impose that the radial derivative vanishes at $\rho' = 1$, then we get $j = \frac{n}{2} + 2s + 1$. We see from (3.21) that the B-branes with $\widehat{j} < k/4$ have open string states with these two sets of eigenvalues. It is tempting to interpret these two possible boundary condition as the ones that would result from taking wavefunctions that are either symmetric or antisymmetric under the exchange of the two disks that form the B-branes.

In the particular case of $\widehat{j} = k/4$ where we have a brane wrapping the whole manifold, we have only one copy of the brane, and therefore we have only one set of eigenfunctions, the one with $j = n/2 + 2\mathbb{Z}$, as we can see from (3.24).

As in the case of closed strings, we get $|n/2| \leq j$. Parafermions with $n$ outside this range are interpreted as massive string states.

## 5. D-branes in the superparafermion theory

### 5.1 Qualitative description

The $\mathcal{N}=1$ supersymmetric version of parafermions turns out to have also $\mathcal{N}=2$ supersymmetry. In fact they are the $\mathcal{N}=2$ minimal models. Some aspects of these models are summarized in appendix E.

We can think of the model as given by the coset

$$\frac{\mathrm{SU}(2)_k \times \mathrm{U}(1)_2}{\mathrm{U}(1)_{k+2}}\,. \tag{5.1}$$

The states in this model are parametrized by $(j,n,s)$ where $j$ is an $\mathrm{SU}(2)_k$ spin, $n$ is a $\mathrm{U}(1)_{k+2}$ label (i.e. it is an integer $\bmod 2(k+2)$), and $s$ is a $\mathrm{U}(1)_2$ label. The labels should be such that $2j+n+s$ is even. The labels $(j,n,s)$ and $(\frac{k}{2}-j, n+k+2, s+2)$ describe the same state.[4] We denote the set of distinct labels by $SPF(k)$. States with $s$ even are in the NS-NS sector and states with $s$ odd are in the RR sector. If we consider the diagonal modular invariant of (5.1), we do not have NS-R sectors. If we are interested in the superstring, we would have to include them. In this section we will restrict the discussion to the model (5.1) with the diagonal modular invariant.

The theory has a discrete symmetry $G$ (see appendix E) generated by $g_1$ and $g_2$ under which the states transform as

$$\begin{aligned} g_1\Psi_{j,n,s} &= e^{2\pi i\left(\frac{n}{2k+4}-\frac{s}{4}\right)}\Psi_{j,n,s}\,,\\ g_2\Psi_{j,n,s} &= (-1)^s\Psi_{j,n,s}\,. \end{aligned} \tag{5.2}$$

Of particular interest to us is the $H=\mathbb{Z}_{k+2}\times\mathbb{Z}_2$ subgroup generated by $g_1^2 g_2$ and $g_2$.

There are $2(k+2)(k+1)$ Ishibashi states and the same number of Cardy states labeled by $(\widehat{j},\widehat{n},\widehat{s})$. These states satisfy A-type boundary conditions. Cardy states with $\widehat{s}=0,2$ obey the boundary conditions[5] $G^\pm=\overline{G}^\mp$, $J=-\overline{J}$. We call these even A-boundary states. Their expansion contains both NS-NS and RR Ishibashi states. The Cardy states with $\widehat{s}=\pm1$ satisfy $G^\pm=-\overline{G}^\mp$. We call these odd A-boundary states. Both types of Cardy states leave unbroken a chiral algebra isomorphic to that of (5.1).

As a sigma model this is an $\mathcal{N}=2$ sigma model whose target space is the disk with our familiar metric with fermions in the tangent space. Using a chiral redefinition these fermions can be made free and can be bosonized to a string scale circle. In terms of the coset (5.1), this circle represents the $\mathrm{U}(1)_2$ factor. The full symmetry group of the theory $G$ acts on the disk times the circle. We would like to have a geometric description of the D-branes on the disk. To do that we have to

[4] We sometimes call this change in labels "spectral flow". This should not be confused with the spectral flow of the $\mathcal{N}=2$ theory that changes NS to R states.

[5] We use open string notation. In the closed string channel there is a factor of $i$ in the first equation and a minus sign in the second.

project the circle onto the disk. Clearly, there is a slight ambiguity in doing that. We find it convenient to project it such that a $\mathbb{Z}_{2k+4} \subset G$ acts geometrically by rotations. Then there are $2k+4$ special points at the boundary of the disk. We break them to two groups: the even points and the odd points.

The $(k+2)(k+1)$ even Cardy states are interpreted as oriented D1-branes stretched between the even points, and the $(k+2)(k+1)$ odd Cardy states are interpreted as oriented D1-branes stretched between the odd points. Unlike the bosonic problem, there are no D0-branes at the special points. More precisely, the geometrical interpretation of A-type Cardy states is that they are oriented straight lines connecting angles $\phi_i$ to $\phi_f$ on the disk. For a state labeled by $(\widehat{j}, \widehat{n}, \widehat{s})$ we have

$$\phi_i = (\widehat{n} - 2\widehat{j} - 1)\frac{\pi}{(k+2)}, \qquad \phi_f = (\widehat{n} + 2\widehat{j} + 1)\frac{\pi}{(k+2)} \tag{5.3}$$

for $\widehat{s} = 0, -1$. For $\widehat{s} = +1, 2$ the state is oriented from $\phi_f$ to $\phi_i$. It is a pleasant exercise to check that this is compatible with the state-identification.[6] Thus, even type A-branes connect odd multiples of $\pi/(k+2)$ around the circle, while odd type A-branes connect even multiples of $\pi/(k+2)$.

The masses of these states are $M_{\widehat{j}} \sim \sin\frac{\pi(2\widehat{j}+1)}{k+2}$. We note that unlike the bosonic problem, here the expression for the mass is exactly the length of the D1-brane and there is no finite shift of order $1/k$ or $1/\widehat{j}$. In [28] very similar branes were studied using the massive Landau-Ginsburg description and some topological quantities were computed. Their results are very reminiscent of ours, but the precise relation between them remains elusive.

We could generalize these boundary conditions and interpolate between these states by considering $G^{\pm} = e^{\pm i\alpha}\overline{G}^{\mp}$. These more general boundary conditions break $N=1$ supersymmetry but preserve all other consistency conditions. For each of the D1-branes mentioned above there is an $S^1$ moduli space of D-branes parametrized by $\alpha$. This $S^1$ is the circle of the bosonized fermions corresponding to the $\mathrm{U}(1)_2$ in (5.1). Two points on this moduli space are special because they preserve the chiral algebra of (5.1) and a fixed $N=1$ subalgebra. This is the familiar spectral flow interpolation between NS and R states. The open strings between these Cardy states would be in representations of the $\mathcal{N}=2$ spectral flowed algebra. The open string spectrum between a brane with any $\alpha$ and itself preserves a full $\mathcal{N}=2$ algebra and therefore contains a massless state in the open string sector associated to the U(1) current in the $\mathcal{N}=2$ algebra, $J_{-1}$. This is the modulus associated to changes in $\alpha$. We will not discuss this further here.

Let us now mod out this theory by the $\mathbb{Z}_2$ symmetry generated by $g_2$ in (5.2). This projects out all the RR closed string states. The twisted sector states are also RR states with the opposite fermion number. The partition function of the

[6]If we choose the branch $-\pi \leq \phi \leq \pi$ for our angles then we should choose the fundamental domain $2j-k-1 \leq n \leq k-2j+1$ for the labels on the representations.

original model is $Z_{\rm diag} = \frac{1}{2}\sum_{j,n,s} |\chi_{j,n,s}|^2$ and is related to the partition function of the quotient theory by

$$Z' = \frac{1}{2}\sum_{j,n,s} \chi_{j,n,s}\overline{\chi}_{j,n,-s} = Z_{\rm diag} - (k+1)\,. \qquad (5.4)$$

The additive constant comes from the Ramond sector states with $L_0 = c/24$. Indeed we have $\chi_{j,2j+1,1} = \chi_{j,2j+1,-1}+1$ (together with the charge conjugate equation). Since the partition functions are distinct the theories are not equivalent. In fact, the D1-branes we discussed above become unoriented. One way to see that is to note that we no longer have the primary RR states which are annihilated by $J_0 - \overline{J}_0$ (recall that while the boundary condition in the open string channel is $J = -\overline{J}$, it is $J = \overline{J}$ in the closed string channel). Therefore, in the orbifold theory, the even Cardy states with $\widehat{s} = 0$ and $\widehat{s} = 2$ are the same and the odd Cardy states with $\widehat{s} = \pm 1$ are also the same.

We can now further mod out this model by $\mathbb{Z}_{k+2}$; i.e. mod out the original model by $H = \mathbb{Z}_{k+2}\times\mathbb{Z}_2$. Now the partition function is $\frac{1}{2}\sum \chi_{j,n,s}\chi_{j,-n,-s}$ which is equal to $Z$. This model is T-dual to the original one. The D1-A-branes in this model are determined in a way similar to the bosonic problem. There is one D1-brane for each value of $\widehat{j} = 0, 1/2, \ldots, \frac{1}{2}[\frac{k-1}{2}]$ with mass $M_{\widehat{j}}$, and for even $k$ two more D1-branes going through the center of the orbifold with mass $\frac{1}{2}M_{\widehat{j}=k/4}$. Since this model is T-dual to the original model (up to a rescaling of $g_s(0)$ by $\sqrt{2(k+2)}$), the super-parafermion theory should also have B-branes labeled by $\widehat{j} = 0, 1/2, \ldots, \frac{1}{2}[\frac{k-1}{2}]$ with masses $\sqrt{2(k+2)}M_{\widehat{j}}$ and for even $k$ two more B-branes covering the space with mass $\frac{1}{2}\sqrt{2(k+2)}M_{\widehat{j}=k/4}$. These are D2-branes at the center of the disk, since the B-brane with $\widehat{j} = 0$ is small we can also refer to it as a D0-brane. Most of these B-branes are unoriented; i.e. they are not accompanied by anti-B-branes with opposite quantum numbers. The two B-branes with $\widehat{j} = k/4$ can be thought of as oriented because they transform to one another by the orbifold $\mathbb{Z}_2$.

Once we have learned that these D2-branes are present in the super-parafermion theory, we can find them also in its $\mathbb{Z}_2$ orbifold. Each of the B-branes with $\widehat{j} = 0, 1/2, \ldots, \frac{1}{2}[\frac{k-1}{2}]$ leads to a D2-B-brane and an anti-D2-B-brane, while the two D2-branes with $\widehat{j} = k/4$ lead to a single D2-brane with that value of $\widehat{j}$.

The original superparafermion theory and its $\mathbb{Z}_2$ orbifold are T-dual to two other models (their $\mathbb{Z}_{k+2}\times\mathbb{Z}_2$ orbifolds) in which the D1-A-branes and the D2-B-branes are interchanged.

These D2-B-branes satisfy the boundary conditions $J = \overline{J}$, $G^\pm = \pm\overline{G}^\pm$ (the sign in the latter expression depends on whether the Cardy state is even ($\widehat{s} = 0, 2$) or odd ($\widehat{s} = \pm 1$)). They preserve an $\mathcal{N} = 2$ subalgebra which differs from that which is preserved by the D1-A-branes. Like the A-branes, they are also part of a one parameter family of D-branes with the boundary conditions $G^\pm = e^{\pm i\beta}\overline{G}^\pm$. We can interpret the parameters $\alpha$ and $\beta$ as a circle (of $\mathrm{U}(1)_2$ in (5.1)) and its dual. The A-branes and the B-branes are D0-branes and D1-branes with respect to this circle.

### 5.2 More quantitative description

The $S$ matrix of the model is

$$S^{SPF \quad j'n's'}_{jns} = 2S_j^{j'} S_n^{n'} S_s^{s'} = \frac{1}{\sqrt{2(k+2)}} S_j^{\ j'} e^{\frac{i\pi nn'}{k+2}} e^{-\frac{i\pi ss'}{2}}, \tag{5.5}$$

where $S_j^{j'}$, $S_n^{n'}$ and $S_s^{s'}$ are the $S$ matrices of $\mathrm{SU}(2)_k$, $\mathrm{U}(1)_{k+2}$ and $\mathrm{U}(1)_2$, respectively. The factor of 2 arises due to spectral flow identification.

**A boundary states** The obvious Cardy states are

$$|A, \widehat{j}\widehat{n}\widehat{s}\rangle_C = \sum_{(jns)\in SPF(k)} \frac{S^{SPF \quad jns}_{\widehat{j}\widehat{n}\widehat{s}}}{\sqrt{S^{SPF \quad jns}_{000}}} |A, jns\rangle\rangle\,. \tag{5.6}$$

The Ishibashi states have the same left and right $s, n$. This implies that in the closed string channel we have $J = \overline{J}$ and therefore $G^{\pm} = i\overline{G}^{\mp}$ or $G^{\pm} = -i\overline{G}^{\mp}$.[7]

Then the open string partition function is obtained as

$${}_C\langle A, \widehat{j}\widehat{n}\widehat{s}| q_c^{L_0 - \frac{c}{24}} |A, \widehat{j}'\widehat{n}'\widehat{s}'\rangle_C = \sum_{jns\in SPF(k)} N^{jns}_{\widehat{j},-\widehat{n},-\widehat{s};\widehat{j}'\widehat{n}'\widehat{s}'} \chi_{jns}(q_o)\,, \tag{5.7}$$

where we sum only over distinct states. The fusion rules are

$$N^{jns}_{\widehat{j},-\widehat{n},-\widehat{s};\widehat{j}'\widehat{n}'\widehat{s}'} = N^{j}_{\widehat{j},\widehat{j}'} \delta_{n-\widehat{n}+\widehat{n}'} \delta_{s-\widehat{s}+\widehat{s}'} + N^{\frac{k}{2}-j}_{\widehat{j},\widehat{j}'} \delta_{n+k+2-\widehat{n}+\widehat{n}'} \delta_{s+2-\widehat{s}+\widehat{s}'}\,, \tag{5.8}$$

where the delta functions are periodic delta functions with the obvious periods, in other words $\delta_n$ had period $2(k+2)$ and $\delta_s$ period 4 (the index distinguishes the periods). Using (5.8) it is convenient to write (5.7) as

$${}_C\langle A, \widehat{j}\widehat{n}\widehat{s}| q_c^{L_0 - \frac{c}{24}} |A, \widehat{j}'\widehat{n}'\widehat{s}'\rangle_C = \sum_j N^{j}_{\widehat{j}\widehat{j}'} \chi_{j,\widehat{n}-\widehat{n}',\widehat{s}-\widehat{s}'}(q_o)\,. \tag{5.9}$$

Let us now comment on the geometrical interpretation of (5.9). First, we must bear in mind that the bra ${}_C\langle A, \widehat{j}\widehat{n}\widehat{s}|$ represents a brane with the *opposite* orientation to the brane $|A, \widehat{j}\widehat{n}\widehat{s}\rangle_C$. Thus, the open string spectrum on a single brane is computed from (5.9) with $\widehat{j} = \widehat{j}'$ and $\widehat{n} = \widehat{n}'$, but $\widehat{s} - \widehat{s}' = 2$. Using eq. (E.12) we see that there is no tachyon on the A-brane. Hence, our A-branes are stable branes. Switching to $\widehat{s} = \widehat{s}'$ we find many tachyons. This is expected for overlapping branes and anti-branes.

Next let us study the open string spectrum for strings connecting different branes. There is an interesting interplay between the existence of tachyons in the open string spectrum and the geometrical intersection of the two branes. Roughly speaking,

[7] Notice that this is consistent with the commutation relations of the current algebra once we take into account that these equations only hold when acting on the boundary state i.e. $[J, G^{\pm}]|B\rangle = [i\overline{G}^{\mp}, \overline{J}]|B\rangle$, i.e. the order of the operators is reversed.

when two even-type or odd-type A-branes intersect, or are close together, we find instabilities. When the branes do not intersect and are more than a string length apart the configuration is stable. If even-type and odd-type branes intersect the open string is in the Ramond sector and there are no tachyons.

More precisely, suppose the brane $(\widehat{j},\widehat{n},\widehat{s})$ is a straight line between angles $[\phi_i,\phi_f]$ given in (5.3). Denoting the analogous angles for $(\widehat{j}',\widehat{n}',\widehat{s}')$ by $[\phi_i',\phi_f']$ the straight lines intersect iff

$$\phi_i \leq \phi_i' \leq \phi_f \leq \phi_f' \,, \qquad \text{or} \qquad \phi_i' \leq \phi_i \leq \phi_f' \leq \phi_f \,. \tag{5.10}$$

Let us consider the first case, for definiteness. Then the three inequalities imply that $|2\widehat{j}-2\widehat{j}'| \leq \widehat{n}-\widehat{n}' \leq 2\widehat{j}+2\widehat{j}'+2$. One easily checks that if $\widehat{s}-\widehat{s}'=0$ then there is at least one tachyon in the open string sector (and often many more). If $\widehat{s}-\widehat{s}'=2$ there is no tachyon. The instability signaled by these tachyons is easy to interpret geometrically. When two branes intersect they tend to break and form two shorter branes which do not intersect. This breaking must be consistent with the orientation. When one brane ends where another brane starts the instability is toward the merging of these two branes into a single shorter brane.[8]

Now let us consider the case where the branes do not intersect. In this case either $\phi_i < \phi_f < \phi_i' < \phi_f'$ or $\phi_i < \phi_i' < \phi_f' < \phi_f$ (up to exchange of $\phi$ with $\phi'$). In the first case there are no tachyons. In the second case there can be tachyons even when the branes do not intersect. Nevertheless, one can check using eq. (E.12) that this happens only when the non-intersecting branes are separated by a distance of order the string scale, in the string metric.

The decays mentioned above suggest that one can introduce a lattice of conserved charges. For the even A-type states we have $\mathbb{Z}^{k+2}/\mathbb{Z} \cong \mathbb{Z}^{k+1}$, and for the odd A-type states another copy of the same lattice. The quotient by $\mathbb{Z}$ accounts for the fact that a ring of A-type states can shrink into the disk.[9] There are interesting relations of this lattice to equivariant K-theory and to the algebra of BPS states, but we will leave this for another occasion.

**B-boundary states** We can similarly find the B-branes. Let us first find the branes for generic $\widehat{j}$ and then discuss the branes with $\widehat{j}=k/4$ for even $k$.

We can build the generic branes as outlined above. We start with branes in a PF' model with partition function $\sum \chi_{j,n,s}\overline{\chi}_{j,-n,-s}$. Then we quotient these branes by $\mathbb{Z}_{k+2}\times\mathbb{Z}_2$ to get branes in the original model, i.e. $PF = PF'/(\mathbb{Z}_{k+2}\times\mathbb{Z}_2)$. The

[8] The two intersecting branes define diagonals on a quadrilateral. Denoting by $\ell_{\phi_1,\phi_2}$ the euclidean length of a straight line between angles $\phi_1,\phi_2$ it follows from the triangle inequality that we have $\ell_{\phi_i'\phi_i}+\ell_{\phi_f\phi_f'} \leq \ell_{\phi_i'\phi_f'}+\ell_{\phi_i\phi_f}$ as well as $\ell_{\phi_i\phi_f'}+\ell_{\phi_i'\phi_f} \leq \ell_{\phi_i'\phi_f'}+\ell_{\phi_i\phi_f}$. Thus there is non-zero phase space for the decay channel in both orientations. The presence or absence of tachyons dictates whether this decay takes place at string tree level or nonperturbatively.

[9] We thank M. Douglas for a useful discussion on this.

Ishibashi states of the $PF'$ model are $|B, j, n, s; j, -n, -s\rangle\rangle$. The only Ishibashi states that exist also in the $PF$ model are the ones with $n = 0, k+2$, $s = 0, 2$ and any $j$. Using the spectral flow identification it is enough to consider only $n = 0$ with $s = 0, 2$. For even $k$ we can also have the states $|B, k/4, \frac{k+2}{2}, \pm 1; k/4, -\frac{k+2}{2}, \mp 1\rangle\rangle$. The label B reminds us that these states obey different boundary conditions for the supercurrents so that they are not identical to the A Ishibashi states.

If we start with Cardy states similar to (5.6) in the $PF'$ model and then we add all the images under $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ we get a state that is invariant under $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ and is therefore in the orbifold theory. The resulting state contains Ishibashi states with only $s = 0, 2$ and $n = 0$.

$$|B, \widehat{j}, \widehat{n}, \widehat{s}\rangle_C = \sqrt{2(k+2)} \sum_{j \in \mathbb{Z}, s=0,2} \frac{S^{SPF}{}_{\widehat{j},\widehat{n},\widehat{s}}{}^{j0s}}{\sqrt{S^{SPF}{}_{000}{}^{j0s}}} |B, j, 0, s; j, 0, -s\rangle\rangle \,. \tag{5.11}$$

The factor of $\sqrt{2(k+2)}$ arises from normalizing the open string channel. Alternatively, it can also be viewed as a standard factor arising whenever we do orbifolds. Note the weak dependence on the labels $\widehat{n}$,$\widehat{s}$. They simply serve to enforce the selection rule. Since we are summing only over $s$ even, only the values of $\widehat{s} = 0, 1$, which correspond to $G = \pm\overline{G}$ on the boundary, produce distinct states. So the total number of states is $2[k/2]+1$ (not counting the factor of 2 coming from the two signs in the $G$ boundary condition). These come from $\widehat{j} = 0, 1/2, \ldots, [k/2]/2$. (We will later see that for $k$ even the last of them is reducible). These branes are not oriented because there are no RR Ishibashi states.

We can now compute the open string spectrum between these branes

$$\begin{aligned} {}_C\langle B, \widehat{j}, \widehat{n}, \widehat{s}| q_c^{L_0 - \frac{c}{24}} |B, \widehat{j}', \widehat{n}', \widehat{s}'\rangle_C = & \sum_{(j,n,s) \in SPF(k)} \left( N_{\widehat{j}\widehat{j}'}{}^{j} + N_{\widehat{j}\widehat{j}'}{}^{\frac{k}{2}-j} \right) \frac{(1 + (-1)^{\widehat{s}' - \widehat{s} - s})}{2} \times \\ & \times\ \chi_{(j,n,s)}(q_o) \,. \end{aligned} \tag{5.12}$$

We see that the identity open string state appears once in the diagonal matrix elements except for the case $\widehat{j} = k/4$ for $k$ even when it appears twice. This is an indication that the B-brane with $\widehat{j} = k/4$ is reducible.

Let us now study this case in more detail. From now on $k$ is even. So there is an extra Ishibashi state that we did not use in (5.11). In order to keep notation simple we define the state $|B, \widehat{s}\rangle_C$ to be the state in (5.11) with $\widehat{j} = k/4$. The new states are

$$\begin{aligned} \left| B, \frac{k}{4} \widehat{s} \right\rangle_C = \frac{1}{2} \Bigg( & |B, \widehat{s}\rangle_C + \sqrt{k+2}\, e^{-i\frac{\pi \widehat{s}^2}{2}} \times \\ & \times \sum_{s=1,-1} e^{-i\frac{\pi \widehat{s} s}{2}} \left| B, \frac{k}{4}, \frac{k+2}{2}, s; \frac{k}{4}, -\frac{k+2}{2}, -s \right\rangle\!\!\right\rangle \Bigg) \,, \end{aligned} \tag{5.13}$$

where the coefficient of the extra state was fixed by demanding a reasonable open string partition function. The factor involving $\widehat{s}^2$ is necessary in order to get some $i's$ appearing appropriately. There are four distinct states labeled by $\widehat{s}$. The branes (5.13) are oriented because there are RR Ishibashi states.

Then the open string partition function between the states (5.13) is

$$
{}_C\left\langle B\frac{k}{4}\widehat{s}\right| q_c^{L_0-\frac{c}{24}} \left|B\frac{k}{4}\widehat{s}'\right\rangle_C = \sum_{n,j} \chi_{j,n,\widehat{s}'-\widehat{s}} \frac{1}{2} \times \\
\times \left[1 + (-1)^{(\widehat{s}'^2+\widehat{s}'-\widehat{s}^2-\widehat{s})/2} (-1)^{(2j+n-\widehat{s}+\widehat{s}')/2}\right]. \quad (5.14)
$$

The sum is over $j, n$ such that $j \in \mathbb{Z}$ and $n + \widehat{s}' - \widehat{s} = 0 \bmod 2$. All states have coefficient one. Notice that the quantities in the exponent are integers due to the selection rule. Putting $\widehat{s} = \widehat{s}' + 2$ we see that these branes are stable.

We can similarly compute the open string spectrum between one of these branes and the other B-branes. In this case the new Ishibashi states in (5.13) do not contribute and we just get half the contribution in (5.12), but with $\widehat{j} = k/4$. It is easy to check that for this case all states appear twice in (5.12), so that when we take half of (5.12), with $\widehat{j} = k/4$, we get a sensible open string spectrum. This open string spectrum only depends on the value of $\widehat{s}$ in (5.13) modulo 2, which implies that we get the same spectrum for the brane and the antibrane at $\widehat{j} = k/4$.

The open string spectrum between the A-type and B-type branes is computed in appendix B.

### 5.3 Remarks concerning the Witten index

It is interesting to note that the U(1) current in the $\mathcal{N} = 2$ algebra is

$$
J = \frac{s}{2} - \frac{n}{k+2}. \quad (5.15)
$$

The NS sector chiral primaries are given by $\phi_{l,-2l,0}$ (and antichiral primaries given by $\phi_{l,2l,0}$), with $l = 0, 1/2, \ldots, k/2$. So there are $k+1$ chiral primaries. They are also equivalently given by the labels related by spectral flow to these. Spectral flow from the NS to the RR sector is given by the change $(n, s) \to (n+1, s+1)$ in the labels of the field. In the NS sector the spectrum of $J$ for chiral primaries is bounded as $0 \le J \le c/6$. The chiral primaries in the R sector are $\phi_{l,2l+1,1}$.

It is interesting for some purposes to compute the Witten index $\mathrm{Tr}_R(-1)^F q_o^{L_0}$ in the open string R sector. In order to do this computation we need to have a boundary state with $G = \overline{G}$ boundary condition on one side and $G = -\overline{G}$ boundary condition on the other side of the strip. This will ensure that the open string has R boundary conditions which preserve supersymmetry. Given the Cardy state $|A, \widehat{j}, \widehat{n}, \widehat{s}\rangle_C$ we produce a state with opposite boundary conditions by considering $|A, \widehat{j}, \widehat{n}+1, \widehat{s}+1\rangle_C$. Notice that we need to shift $\widehat{n}$ due to the selection rule. We want to think about them, however, as describing the same brane. The Witten index was computed in [28, 29], we just reproduce their calculation and include the index between A-branes and the new B-brane.

We also need to introduce the factor of $(-1)^F$ in the computation. This factor will imply that only RR states propagate in the closed string channel. We know that if we have the Cardy state with label $\widehat{s}$, then we reverse the sign of the RR sector in the closed string channel by shifting $\widehat{s} \to \widehat{s}+2$. Then we conclude that the Witten index is

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}_{\alpha\beta}}(-1)^F &= {}_C\langle A\widehat{j},\widehat{n},\widehat{s}|q_c^{L_0-\frac{c}{24}}|A\widehat{j}',\widehat{n}',\widehat{s}'\rangle_C - \\ &\quad - {}_C\langle A\widehat{j},\widehat{n},\widehat{s}|q_c^{L_0-\frac{c}{24}}|A,\widehat{j}',\widehat{n}',\widehat{s}'+2\rangle_C\,, \end{aligned} \tag{5.16}$$

where $\widehat{s}-\widehat{s}' = \pm 1 \mod (4)$ in order for the open string Hilbert space $\mathcal{H}_{\alpha\beta}$ corresponding to boundary conditions $\alpha\beta$ to be in the Ramond sector.[10]

Using (5.16) and (5.9) we find that the Witten index between two A-branes (with $\widehat{s}-\widehat{s}' = \pm 1 \mod (4)$) is

$$\mathrm{Tr}(-1)^F = (-1)^{\frac{\widehat{s}-\widehat{s}'-1}{2}} \sum_j N^j_{\widehat{j},\widehat{j}'}\left(\chi_{j,\widehat{n}-\widehat{n}',1} - \chi_{j,\widehat{n}-\widehat{n}',-1}\right). \tag{5.17}$$

Chiral primaries can only appear in the R sector if $2j = \widehat{n}-\widehat{n}'-1$ from the first term or if $k-2j = k+2+\widehat{n}-\widehat{n}'-1$ in the second term. In order to make formulas more transparent we take $\widehat{s}=1$, $\widehat{s}'=0$, and define $\widehat{n} = \widetilde{n}+1$. So we think of $(\widehat{j},\widetilde{n})$ and $(\widehat{j}',\widehat{n}')$ as the labels of the two states. All together we get

$$\mathrm{Tr}(-1)^F = N^{\widetilde{n}-\widehat{n}'}_{\widehat{j}\widehat{j}'}\,, \tag{5.18}$$

where, as in [29] we defined $N^{-j-1}_{j,j'} = -N^j_{j,j'}$ and $N^{-1/2}_{j,j'} = N^{\frac{k}{2}+\frac{1}{2}}_{j,j'} = 0$. Notice that only one of the two terms in (5.17) is non zero. Using (5.10) one can show that the condition for the existence of a chiral primary is precisely the condition for the D1-branes to have an intersection, and that moreover (5.17) is simply the oriented intersection number of the two lines. This is, once again, strongly reminiscent of the result of [28].

Note that the Witten index between any of the A states and the unoriented B states is zero, as well as between any of the unoriented B states, due to the fact that the B boundary states have no RR sector states.

We now compute the Witten index between the oriented B- branes with $\widehat{j} = k/4$ and the $A$ branes:

$${}_C\left\langle B\frac{k}{4}\widehat{s}'\right|q_c^{L_0-c/24}\left(|A\widehat{j},\widehat{n},\widehat{s}\rangle_C - |A\widehat{j},\widehat{n},\widehat{s}+2\rangle_C\right). \tag{5.19}$$

[10]Notice that there is no factor of 1/2 in (5.16) because the overlaps of Ishibashi states have definite fermion number, they are really $\mathrm{Tr}[\frac{(1\pm(-1)^F)}{2}q_c^{L_0}]$. So in (5.16) we extract the term with $(-1)^F$.

This can be written as

$$\sin\left[\frac{\pi}{2}(2\widehat{j}+1)\right] e^{i\pi(\widehat{s}')^2/2}\left(e^{\frac{i\pi}{2}(\widehat{s}'+\widehat{n}-\widehat{s})}f_+ + e^{\frac{i\pi}{2}(-\widehat{s}'-\widehat{n}-\widehat{s})}f_-\right), \tag{5.20}$$

where $f_+, f_-$ are the overlaps

$$\begin{aligned} f_+ &= \left\langle\left\langle B\frac{k}{4}, \frac{k+2}{2}, 1; -\frac{k+2}{2}, -1 \right| q_c^{L_0-\frac{c}{24}} \left| A\frac{k}{4}, \frac{k+2}{2}, 1\right\rangle\right\rangle, \\ f_- &= \left\langle\left\langle B\frac{k}{4}, \frac{k+2}{2}, -1; -\frac{k+2}{2}, +1 \right| q_c^{L_0-\frac{c}{24}} \left| A\frac{k}{4}, \frac{k+2}{2}, -1\right\rangle\right\rangle. \end{aligned} \tag{5.21}$$

These overlaps are related to conformal blocks of one-point functions of simple currents on the torus, and as such have been much studied (see, for examples, [30, 31, 32]). However, the only key fact we need is that the result must be $q_c$-independent and therefore we can take the $q_c \to 0$ limit and only keep the terms with the RR groundstates.[11] The resulting overlap is then immediately evaluated to be:

$$\begin{aligned} {}_C\left\langle B\frac{k}{4}\widehat{s}'\right| q^H \left(|A\widehat{j}\widehat{n}\widehat{s}\rangle_C - |A\widehat{j}\widehat{n}\widehat{s}+2\rangle_C\right) &= \\ &= \begin{cases} 0\,, & \widehat{j} \in \frac{1}{2}+\mathbb{Z}\,, \\ (-1)^{\widehat{s}'(\widehat{s}'+1)/2}(-1)^{(2\widehat{j}+\widehat{n}-\widehat{s})/2}\,, & \widehat{j}\in\mathbb{Z}\,. \end{cases} \end{aligned} \tag{5.22}$$

The Witten index of boundary states can usually be interpreted in terms of intersection theory. The expression (5.22) may likewise be identified with the intersection between the lines associated with A-branes with a signed sum of points on the boundary of the disk. For example, the intersection number (5.22) is reproduced by the intersection number of the A-branes with the signed sum of points $(-1)^{\widehat{s}'(\widehat{s}'+1)/2}(P_0+P_1)$, where $P_0$ consists of the sum of points at the boundary of the form $4n\frac{\pi}{k+2}$ and $P_1$ is the sum of points at the boundary of the form $(4n+1)\frac{\pi}{k+2}$. This interpretation suggests the possibility that there might be a non-perturbative instability of the special disk-filling B-brane to a collection of oriented D0-branes on the boundary of the disk.

## 6. D-branes in SU(2) and Lens spaces

In this section we will construct some new branes for the SU(2) WZW model, these are branes which are analogous to the B-branes we described for parafermions. In the case of parafermions, it was useful to view the model as a $\mathbb{Z}_k$ quotient, up to a T-duality. The same idea will be useful in the SU(2) case. So we first start reviewing Lens spaces, which are interesting quotients of SU(2), and then we will proceed to discuss the branes.

[11] Technically only $f_+$ has a contribution from the RR groundstate, and in fact $f_+ = 1 + f_-$ where $f_-$ is a non-trivial function of $q_c$. There is a cancellation in the two terms in (5.20).

## 6.1 Lens spaces

The Lens space is $S^3/\mathbb{Z}_{k_1}$ where the $\mathbb{Z}_{k_1}$ is a discrete subgroup acting on the left (as opposed to the U(1) discussed thus far). In terms of the coordinates in (2.18) it corresponds to the identification $\chi \sim \chi + \frac{4\pi}{k_1}$. This is a quotient without fixed points. The Lens space is completely non singular. In terms of the SU(2) WZW model this is the orbifold $\mathrm{SU}(2)/\mathbb{Z}^L_{k_1}$ where $\mathbb{Z}^L_{k_1}$ is embedded in the left U(1). In order for this theory to be consistent, the level $k$ of the SU(2) covering theory should be of the form $k = k_1k_2$ where $k_1, k_2 \in \mathbb{Z}$. This can be understood as quantization of three form $H$ in the coset theory. In string theory this theory arises as the transverse geometry in coincident $A_{k_1-1}$ singularities and NS 5-branes.

The partition function for this theory was studied in [33], where it was written using parafermions. Using (2.9) the partition function of $\mathrm{SU}(2)_k/\mathbb{Z}^L_{k_1}$ is

$$\begin{aligned} Z &= \sum_j \sum_{\substack{n+n'=0 \mod 2k_1 \\ n-n'=0 \mod 2k_2}} \sum_{\overline{n}=-(k-1)/2}^{k/2} \chi^{PF}_{jn}(q)\chi^{\mathrm{U}(1)}_{n'}(q)\chi^{PF}_{j\overline{n}}(\overline{q})\chi^{\mathrm{U}(1)}_{\overline{n}}(\overline{q}) \\ &= \sum_j \sum_{\substack{n+n'=0 \mod 2k_1 \\ n-n'=0 \mod 2k_2}} \chi^{PF}_{jn}(q)\chi^{\mathrm{U}(1)}_{n'}(q)\chi^{\mathrm{SU}(2)}_j(\overline{q})\,. \end{aligned} \tag{6.1}$$

We can check that it is modular invariant. Note that in the case of $k_1 = 2$ it gives the SO(3) model (depending on whether $k_2$ is even or odd we get the two SO(3) modular invariants from (6.1)). If $k_1 = 1$ we recover the SU(2) partition function.

We can think of $(n - n')/2$ as the winding number $l_w$ which is defined $\mod k_1$ through $n - n' = 2l_wk_2$, $l_w = 0, 1, \ldots, k_1 - 1$. We could also define a "momentum" through $n + n' = 2l_pk_1$ and $l_p$ is defined $\mod k_2$. We see that the model is invariant under the exchange of $k_1$ and $k_2$ which amounts to a reversal in the sign of $n'$ (or a T-duality in the circle direction). It is interesting to note that, even though the momentum and the winding are not conserved, the charge $n' = (l_pk_1 - l_wk_2)$ is indeed conserved since it is the $\mathrm{U}(1)_L$ charge which is left unbroken by the orbifold.

We see that (6.1) is invariant under interchanging $k_1 \leftrightarrow k_2$, up to a T-duality that reverses the sign of $n'$. Therefore, $\mathrm{SU}(2)_{k_1k_2}/\mathbb{Z}_{k_1} = \mathrm{SU}(2)_{k_1k_2}/\mathbb{Z}_{k_2}$. In particular $\mathrm{SU}(2)_k/\mathbb{Z}_k = \mathrm{SU}(2)_k$.

Note that this theory, given by (6.1), is an asymmetric orbifold. So D-branes in this theory are examples of D-branes in asymmetric orbifolds.

We note that all these theories including the $\mathrm{SU}(2)_k$ theory are rational with respect to the chiral algebra $\mathcal{C} = \mathcal{A}^{PF_k} \times \mathrm{U}(1)_k \subset \mathrm{SU}(2)_k$. The diagonal modular invariant of this algebra is $\sum_{jnn'} |\chi^{PF}_{jn}|^2 |\psi^{\mathrm{U}(1)}_{n'}|^2$. Other partition functions like the diagonal modular invariant of $\mathrm{SU}(2)_k$ or (6.1) are different modular invariants of $\mathcal{C}$. In particular, the chiral algebra of $\mathrm{SU}(2)_k$ can be thought of as the extended chiral algebra of $\mathcal{C}$ which can be denoted as $\mathcal{C}/\mathbb{Z}_k$. Below, we will find this interpretation of the SU(2) theory convenient. All the branes we will consider can be found using the



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Cardy procedure [2] and its generalization in [6, 7, 34] with respect to the algebra $\mathcal{C}$. In doing that one must make sure that the expansion of the boundary states in terms of Ishibashi states includes only closed string states which are present in the theory. Below, we do not follow this algebraic route. Rather, we will construct the branes using a more geometric procedure.

### 6.2 Branes in $\mathrm{SU}(2)_k$ and $\mathrm{SU}(2)_k/\mathbb{Z}_k$

The standard Cardy theory applied to the SU(2) level $k$ WZW model produces a set of Cardy states

$$|A,\widehat{j}\rangle_C = \sum_j \frac{S_{\widehat{j}}^{\ j}}{\sqrt{S_0^{\ j}}}|A,j\rangle\rangle\,. \tag{6.2}$$

These states have been interpreted geometrically as states made out of $2\widehat{j}+1$ D0-branes. These grow into D2-branes which wrap conjugacy classes in the group [11]–[14].

It is convenient to express the SU(2) Ishibashi states as

$$|A,j\rangle\rangle = \sum_{n=1}^{2k} \frac{1+(-1)^{2j+n}}{2}|A,j,n\rangle\rangle_{PF}|A,n\rangle\rangle_{\mathrm{U}(1)}\,. \tag{6.3}$$

When we quotient the theory by $\mathbb{Z}_{k_1}$ we find that not all SU(2) Ishibashi states appear in the quotient theory. More to the point, the Cardy states (6.2) are not invariant under $\mathbb{Z}_{k_1}$. We can form $\mathbb{Z}_{k_1}$ invariant states by adding all $\mathbb{Z}_{k_1}$ images. This automatically projects onto Ishibashi states that are present in the quotient theory and thus produces Cardy states that exist in the Lens space theory.

We now follow the strategy which proved useful above. We consider the case $k_1 = k$ and perform T-duality to the original $\mathrm{SU}(2)_k$ theory. Then we reinterpret these branes as new B-branes in $\mathrm{SU}(2)_k$. We find

$$\begin{aligned}|B,\widehat{j},\eta=\pm1\rangle_C &= \sum_{j=0}^{k/2}\frac{\sqrt{k}S_{\widehat{j}}^{\ j}}{\sqrt{S_{0j}}}\left[\frac{1+(-1)^{2j}}{2}|Aj0\rangle\rangle_{PF}|B00\rangle\rangle_{\mathrm{U}(1)}+\right.\\ &\qquad\left.+\eta(-1)^{2\widehat{j}}\frac{1+(-1)^{2j+k}}{2}|Ajk\rangle\rangle_{PF}|B,k,-k\rangle\rangle_{\mathrm{U}(1)}\right]\\ &= \sum_{\substack{j=0\\ j\in\mathbb{Z}}}^{k/2}\frac{\sqrt{k}S_{\widehat{j}}^{\ j}}{\sqrt{S_{0j}}}|Aj0\rangle\rangle_{PF}\left[|B00\rangle\rangle_{\mathrm{U}(1)}+\eta|B,k,-k\rangle\rangle_{\mathrm{U}(1)}\right]\\ &= \frac{1}{\sqrt{k}}|B\eta\rangle_C^{\mathrm{U}(1)}\sum_{\widehat{n}=0}^{2k-1}|A\widehat{j}\widehat{n}\rangle_C^{PF}\,.\end{aligned} \tag{6.4}$$

Note the use of the two B-branes in $\mathrm{U}(1)_k$ defined in (3.4). The last line of (6.4) shows that $|B,\widehat{j},\eta\rangle_C = |B,\frac{k}{2}-\widehat{j},\eta\rangle_C$, and therefore the different branes are labeled by $\widehat{j} = 0, 1/2, \ldots, 1/2[k/2]$.

As we mentioned above, we can view the $\mathrm{SU}(2)_k$ model as an orbifold $\mathrm{SU}(2)_k = (\mathrm{U}(1)_k \times PF_k)/\mathbb{Z}_k$. The last line of (6.4) becomes easy to understand, we just take products of Cardy states and we take a superposition that is invariant under $\mathbb{Z}_k$. Similarly, we can express (6.2) as

$$|A, \widehat{j}\rangle_C = \frac{1}{\sqrt{k}} \sum_{\widehat{n}} |A\widehat{j}\widehat{n}\rangle_C^{PF} |A\widehat{n}\rangle_C^{\mathrm{U}(1)} \tag{6.5}$$

which is obvious from the orbifold interpretation but can also be checked explicitly using the expressions for the various factors. It is now straightforward to compute the annulus diagrams

$$\begin{aligned} {}_C\langle B, \widehat{j}\eta | q_c^{L_0 - \frac{c}{24}} | B, \widehat{j}'\eta'\rangle_C &= \sum_j \sum_{n=0}^{2k-1} \sum_{n'=0}^{2k-1} N_{\widehat{j}\widehat{j}'}^{j} \chi_{j,n}(q_o) \psi_{n'}(q_o) \frac{1 + \eta\eta'(-1)^{n'}}{2}\,, \\ {}_C\langle A\widehat{j} | q_c^{L_0 - \frac{c}{24}} | B\widehat{j}'\rangle_C &= \sum_{j,n} N_{\widehat{j}\widehat{j}'}^{j} \chi_{j,n}(q_o) \frac{(q_o)^{1/48}}{\prod_m (1 - (q_o)^{m+\frac{1}{2}})}\,. \end{aligned} \tag{6.6}$$

The geometrical interpretation of the B-type state is that it represents a three-dimensional object corresponding to a "thickened" or "blown-up" D1 string. This can be substantiated by the computation of the shape of the brane at large $k$ (see appendix D) and from the various properties listed in the following comments.

(1) From the mode expansion in (6.6) we can read off the spectrum of the open strings stretched between two B-branes or between a B-brane and the familiar A-branes.

(2) The normalizations in (6.4) can be deduced from the T-duality argument and is such that the coefficients of the various modes in (6.6) are non-negative integers.

(3) The label $\eta$ in the state can be interpreted as a Wilson line around the B-brane. To see that, note that a Wilson line introduces a phase $e^{i\pi a n/k}$ in front of a boundary state $|B, n, -n\rangle\rangle_{\mathrm{U}(1)}$. For the special values $a = 1$ and $n = k$ this corresponds to $\eta = -1$ in (6.4). For generic values of $a$ the boundary state breaks the extended $\mathrm{U}(1)_k$ chiral algebra because a state with momentum $n$ gets a phase different from a state with momentum $n + 2k$. Other values of $a$ are of course good Cardy states, and it is straightforward to describe them. We have to use Ishibashi states and characters of the non-compact U(1) generated by the current instead of those of $\mathrm{U}(1)_k$. This comment follows from the behavior of the circle and is identical to the phenomena we saw in section 3.1.

(4) The various D-brane charges of $|B\widehat{j}\pm\rangle$ can be determined by the overlap with $|A\widehat{j} = 1/2, \widehat{n} = 0\rangle\rangle$. Since this overlap vanishes, all these charges also vanish. Note that this is unlike the situation with the A-branes $|A\widehat{j}\rangle$, which have non-zero D0-brane charge. We should comment, however, that the issue of the charge makes sense only in the context of the superstring.



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(5) The mass of $|B\widehat{j}\eta\rangle$ is determined by the overlap with $|A, j = 0, n = 0\rangle\rangle$. We easily find

$$M(B\widehat{j}) = \sqrt{k} M(A\widehat{j}) \sim k^{3/2} \sin \frac{\pi(2\widehat{j}+1)}{k+2}$$
$$= \begin{cases} k^{3/2} \sin \frac{2\pi\widehat{j}}{k}\,, & \widehat{j} \approx \frac{k}{4} \gg 1\,, \\ k^{1/2}(2\widehat{j}+1)\,, & k \gg \widehat{j} \approx 1\,. \end{cases} \tag{6.7}$$

This fact has the following interpretation. The radius of the target space is $\sqrt{k}$. Therefore the new branes have the same width as the corresponding A-branes but they wrap the target space. They have the form of a disk (as in section 4.2) times a circle of the size of the group.

(6) The factor $\frac{(q_o)^{1/48}}{\prod_m (1-(q_o)^{m+\frac{1}{2}})}$ in the second inner product in (6.6) can be interpreted, as in flat space as due to a single boson with D boundary conditions at the D0-brane and N boundary conditions at the D1-brane.

(7) Notice that since the T-duality that maps $\mathrm{SU}(2)_k/\mathbb{Z}_k$ to $\mathrm{SU}(2)_k$ only changes the U(1) part the states continue to obey A-type boundary conditions for the parafermion part but obey B-type for U(1).

(8) We could also have taken a B state for the parafermions and an A state for the U(1). This would have given states that differ from (6.4) by a rotation in the group.

(9) The diagonal matrix elements ${}_C\langle B, \widehat{j} = k/4, \eta | q_c^{L_0 - \frac{c}{24}} | B, \widehat{j} = k/4, \eta\rangle_C$ (6.6) show that the open string Hilbert spaces on these D-branes include the identity operator twice. Therefore, as in the parafermion theory, the B-branes with $\widehat{j} = k/4$ are reducible and can be written as sums of two branes. We expect, by analogy with the parafermion theory, that these D-branes are stable. Also, we expect them to cover the whole $S^3$ target manifold but to have singularities. We did not explore in detail the properties of these branes to verify these expectations. In order to think about these special branes it is convenient to think of $\mathrm{SU}(2)_k = (\mathrm{U}(1)_k \times PF_k)/\mathbb{Z}_k$, and construct Cardy states that are products of B-branes in PF and A-branes in U(1). If we take the special branes (3.20), multiply by U(1) A-branes and take all the $\mathbb{Z}_k$ images, we get the irreducible branes at $\widehat{j} = k/4$ that we mentioned above.

In order to explore the shape of the B-branes we consider the limit $k \gg \widehat{j} \gg 1$ and examine the open strings which live on them. From (6.6) we see that the spectrum is

$$\Delta_{jnn'} \approx \frac{j(j+1)}{k} - \frac{(n)^2}{4k} + \frac{(n')^2}{4k}\,, \qquad j \in \mathbb{Z}; \qquad n = -2j, -2j+2, \ldots, 2j\,; \qquad n' \in 2\mathbb{Z}\,. \tag{6.8}$$

The quantum number $n'$ is easily interpreted as momentum along a circle. $j'$ and $n$ give us information about the transverse shape of the brane, $n$ can be thought of as angular quantum number around the D1. The simplest case is when $\widehat{j} = 0$. This corresponds to a D1-brane along a maximum circle of $S^3$. It is unstable and the instability corresponds to the D1-brane shrinking. The instability can be seen in (6.6) and comes from the term with $j = 0 = n'$, $n = \pm 2$. The conformal weight of this state is $\Delta = 1 - \frac{1}{k}$, this agrees with the semiclassical expectations for such a D1-brane. It also has the expected angular momentum around the D1. More generally, this is exactly the spectrum discussed on the D2-brane in section 4.2 tensored with a spectrum of open strings on a circle.

We expect to find similar B-branes in the SL(2) WZW model. In fact the D1 in that case would describe D-particles moving in $AdS_3$. The A-branes in SL(2) are those discussed in [35]–[38]. These B-branes in SL(2) would describe D0-branes at the tip of the cigar. It would be interesting to work this out explicitly. It is interesting to note that the B-branes of Liouville theory might be related to the construction in [39], which seems to be somehow localized in the Liouville direction.

Some classical D-string surfaces that seem to look like our B-branes in SU(2) were discussed in the classical analysis of [35], based on the idea of flipping the sign in the SU(2) currents $J^a = -\overline{J}^a$. This boundary condition is not consistent quantum mechanically because it does not respect the *bulk* SU(2) commutation relations (OPEs) for the current algebra (independently of whether they they do or do not preserve the symmetry).[12] Our construction does not translate into a simple boundary condition for the currents. Qualitatively one can think about the boundary conditions in the following way. We can write the SU(2) currents as $J^+ = e^{+i\phi}\psi^+$ and $J^- = e^{-i\phi}\psi^-$. If we think of $\psi^\pm$ as the elementary parafermion fields then we are imposing the boundary conditions $\partial\phi = -\overline{\partial}\,\overline{\phi}$ and $\psi^\pm = \overline{\psi}^\pm$. So we flip the sign of the U(1) boson but we do not do anything to the parafermions. This description is not precisely correct since $\psi^\pm$ are not holomorphic fields in the theory, the proper statement is that the appropriate powers of $\psi^\pm$ that do form holomorphic fields are reflected without any change. This preserves the SU(2) current OPEs.

Classical analysis of D-branes in SU(2) appeared also in [40]. The precise relation of their results to ours is not clear to us.

### 6.3 The U(1) symmetries of the B-branes in SU(2)

The B-branes in SU(2) can be thought of as fat D1-branes. It helps to consider the case of $\widehat{j}$ small to think about this. In the parametrization

$$z_1 = e^{i\phi}\sin\theta\,, \qquad z_2 = e^{i\widetilde{\phi}}\cos\theta\,, \qquad ds^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\widetilde{\phi}^2 \tag{6.9}$$

[12]We thank V. Schomerus for a discussion on this point.

the D1-brane is at $z_1 = 0$.[13] Geometrically it appears to preserve a $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ subgroup of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ isometry of the $S^3$ target space. In the CFT we only saw a single U(1) (and a $\mathbb{Z}_k$). How did this happen?

The $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ symmetries are generated by $J^3_{L,R}$. In the B boundary state the combination generated by $J_L + J_R$ is obviously conserved, while $J^3_L - J^3_R$ is conserved modulo $k$. More geometrically, in the coordinate system (6.9) the brane is concentrated at $z_1 = \theta = 0$. The operator $J_L + J_R$ generates a shift of $\widetilde{\phi}$ and corresponds to translations along the brane. It leads to a conserved charge. The operator $J_L - J_R$ generates a shift of $\phi$ and corresponds to rotations around the brane. It does not lead to a conserved charge. The violation of this charge can be seen in the boundary state (6.4). The term $|B, k, -k\rangle\rangle_{\mathrm{U}(1)}$ in the boundary state shows that the brane can absorb a closed string state with $|j, k, -k\rangle$. Therefore, $J^3_L + J^3_R$ is conserved while $J^3_L - J^3_R$ is violated by $k$ units.

To see this more explicitly we consider a point like closed string at $\theta = \pi/2$ moving along $\phi$. Due to the giant graviton effect [41] it can become a string that winds along the $\widetilde{\phi}$ angle. Let us see whether this is possible. The action for a string with $t = \tau$ and $\widetilde{\phi} = \sigma$ is

$$S = -\int dt \left[\sqrt{k}\cos\theta\sqrt{1 - k\sin^2\theta\dot{\phi}^2} - k\cos^2\theta\dot{\phi}\right], \tag{6.10}$$

where the linear term in $\dot{\phi}$ comes from the $B$-field. The conserved momentum conjugate to $\dot{\phi}$ is

$$J = \frac{\sqrt{k}\cos\theta k\sin^2\theta\dot{\phi}}{\sqrt{1 - k\sin^2\theta\dot{\phi}^2}} + k\cos^2\theta\,, \tag{6.11}$$

and therefore the energy can be expressed in terms of $J$

$$E = \sqrt{k}\sqrt{\frac{(1-\frac{J}{k})^2}{\sin^2\theta} + \left(2\frac{J}{k} - 1\right)}\,. \tag{6.12}$$

It is easy to check that when $J < k$ the minimum energy is obtained for $\sin\theta = 1$ and is $E = J/\sqrt{k}$, while for $J = k$ we obtain $E = \sqrt{k}$ independently of the value of $\sin\theta$. So we can continuously change between a "momentum" mode at $\theta = \pi/2$, and a winding mode, at $\theta = 0$. Indeed at $\theta = 0$ and $J = k$ the string coincides with the brane and it can, therefore, be absorbed by it.

An alternative way to see the violation of the U(1) symmetry is through an instanton calculation, which is essentially the euclidean version of the previous calculation. The relevant instanton is a disk instanton, whose boundary is a D1-brane at $z_1 = 0$ (in the coordinate system (6.9)). We can parametrize the worldsheet by $|z_2| \leq 1$, and choose a worldsheet such that the phase $\phi$ of $z_1 = e^{i\phi}\cos\theta$ is constant.

[13]This brane actually differs from (6.4) by a trivial $\pi$ rotation.

In order to calculate the instanton action we need to find a convenient form of the $B$-field. The $H$-field in the coordinate (6.9) is given by the volume form $H = \frac{k}{2\pi} \sin(2\theta) d\widetilde{\phi} d\phi d\theta$. A possible choice of the $B$-field is $B = \frac{k}{2\pi}\phi \sin(2\theta) d\widetilde{\phi} d\theta$. Therefore, the WZ term of the disk instanton is

$$i \int B = ik\phi \,. \tag{6.13}$$

Clearly, the U(1) symmetry of shifting $\phi$ by a constant is broken to $\mathbb{Z}_k$.

### 6.4 The moduli space of the B-branes in SU(2)

With this understanding of the symmetries we can determine the moduli space of the B-branes. The group that acts on the branes is the isometry $\mathrm{SU}(2) \times \mathrm{SU}(2)$. The geometric picture suggests that a $\mathrm{U}(1)_\phi \times \mathrm{U}(1)_{\widetilde{\phi}}$ subgroup of it, which shifts $\phi$ and $\widetilde{\phi}$ (in (6.9)) by constants leaves the branes invariant. Therefore we might be tempted to guess that the moduli space of these branes is $\frac{\mathrm{SU}(2)\times \mathrm{SU}(2)}{\mathrm{U}(1)_\phi \times \mathrm{U}(1)_{\widetilde{\phi}}}$. This cannot be the right answer because, as we explained above $\mathrm{U}(1)_\phi$ is broken to $\mathbb{Z}_k$, and is not a symmetry of the problem. Therefore, it cannot appear in the denominator of the quotient.

The resolution of the problem is easy when we remember that the B-branes also have another modulus. Since these branes are fat D1-branes, we can turn on Wilson lines along them. These take values in U(1) (we have already encountered some discrete values of these Wilson lines in the parameter $\eta$ in (6.4)) — we add $\oint A = n\rho$ to the worldsheet action where $n$ is the number of times the boundary of the worldsheet winds around the D-brane. Now (6.13) is modified to

$$i \int B + i \oint A = ik\widetilde{\phi} + i\rho \,. \tag{6.14}$$

The $\mathrm{U}(1)_{\widetilde{\phi}}$ symmetry is now restored, if at the same time we also shift the value of the Wilson line $\rho$. We conclude that the moduli space of these D-branes is

$$\frac{\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)_W}{\mathrm{U}(1)_\phi \times \mathrm{U}(1)_{\widetilde{\phi}}} \,, \tag{6.15}$$

where $\mathrm{U}(1)_W$ is the value of the Wilson line and $\mathrm{U}(1)_\phi$ not only shifts $\phi$ (i.e. is in $\mathrm{SU}(2) \times \mathrm{SU}(2)$) but also shifts $\rho$ (i.e. is in $\mathrm{U}(1)_W$).

As a simple consistency check of the answer (6.15), we note that the spectrum of open strings on a B-brane can be read off from (6.6). It includes five dimension one operators which are moduli. One of them is the descendant of the identity operator — the U(1) current. The other four are the parafermion primaries ($j = 0, n = \pm 2$) with U(1) charges $n' = \pm 2$ for the compact boson.

### 6.5 D-branes in more general Lens spaces

In this subsection we explore some of the D-branes in the Lens spaces $S^3/\mathbb{Z}^L_{k_1}$.

The A-branes are easily found by starting with the A-branes in the $\mathrm{SU}(2)_k$ covering theory. In terms of the Cardy states in the covering theory we have to sum over images. These images are rotated in the group and are not simply of the form $|A,\widehat{j}\rangle_C$ we used above. However, in terms of Ishibashi states the $\mathbb{Z}^L_{k_1}$ projection is easily implemented. It states that in $|A,j,n\rangle\rangle_{PF}|A,n,n\rangle\rangle_{\mathrm{U}(1)}$ the value of $n$ must satisfy $n=0 \mod k_1$. Therefore, we are led to study the Cardy states

$$|A\widehat{j}\rangle_C = \sqrt{k_1} \sum_{\substack{j \\ n=0 \mod k_1}} \frac{S_{\widehat{j}j}}{\sqrt{S_{0j}}} \frac{1+(-1)^{2j+n}}{2} |Aj,n\rangle\rangle_{PF}|A,n,n\rangle\rangle_{\mathrm{U}(1)}\,. \tag{6.16}$$

More general A-branes can be found by rotating these branes in the target space $S^3/\mathbb{Z}^L_{k_1}$. For simplicity we will not consider them.

The open string spectrum between these branes is easily computed

$${}_C\langle A\widehat{j}|q^{L_0-\frac{c}{24}}|A\widehat{j}'\rangle_C = \sum_{\substack{j \\ n-n'=0 \mod 2k_2}} N^j_{\widehat{j}\widehat{j}'}\chi^{PF}_{jn}\chi^{\mathrm{U}(1)}_{n'}\,. \tag{6.17}$$

We see that the spectrum includes the usual strings on a single brane, as well as the strings stretched between the different images. From this formula we see that all values of $\widehat{j}$, with $0\le\widehat{j}<k/4$ give distinct branes, since the open string spectrum on them is different. Furthermore, the identity operator appears only once in the open string spectrum on each brane, so these are irreducible branes. These branes can, however, (even in the superstring) decay to each other. For $k$ even the brane with $\widehat{j}=k/4$ is special because the identity operator can appear twice on it. This happens if and only if $k_1$ is even, as in SO(3) ($k_1=2$) [6]. In this case this brane is reducible and should be expressed as a sum of two other branes. It would be nice to explore this case in detail.

The theory also has B-branes

$$|B\widehat{j}\widehat{n}\rangle_C = \sqrt{k_2} \sum_{\substack{j \\ n=0 \mod k_2}} e^{i\pi\widehat{n}n/k} \frac{S_{\widehat{j}j}}{\sqrt{S_{0j}}} \frac{1+(-1)^{2j+n}}{2} |A,j,n\rangle\rangle_{PF}|B,n,-n\rangle\rangle_{\mathrm{U}(1)}\,. \tag{6.18}$$

These are the B-branes of the WZW which are invariant under the $\mathbb{Z}_{k_1}$ orbifold group. The parameter $\widehat{n}$ is interpreted as a Wilson line which takes $2k_1$ possible values. More precisely, as in the rational torus, the Wilson line can have continuous values but our use of the full $\mathrm{U}(1)_k$ chiral algebra led us to only $2k_1$ values. As for the B-branes in SU(2), also here there exists non-trivial mixing between rotations around the branes and the value of the Wilson line.

We can calculate the open string spectrum between these branes:

$$
\begin{aligned}
{}_C\langle B\widehat{j}\widehat{n}|q^{L_0-\frac{c}{24}}|B\widehat{j}'\widehat{n}'\rangle_C &= \sum_{\substack{j \\ n'-n=(\widehat{n}-\widehat{n}') \bmod 2k_1}} N^{j}_{\widehat{j}\widehat{j}'}\chi^{PF}_{jn}\chi^{\mathrm{U}(1)}_{n'}\,, \\
{}_C\langle A\widehat{j}|q^{L_0-\frac{c}{24}}|B\widehat{j}'\rangle_C &= \sum_{jn} N^{j}_{\widehat{j}\widehat{j}'}\chi^{PF}_{jn}\chi^{\mathrm{U}(1)}_{ND}\,.
\end{aligned} \tag{6.19}
$$

Again we see that $\widehat{n}$ is a Wilson line since it changes the momentum quantization condition.

Similarly to the above discussion, for $k_2$ even the B-branes with $\widehat{j} = k/4$ are reducible and should be expressed as sums of two separate branes.

These B-branes on Lens spaces seem relevant for producing the theories described in [42], since they live on a space with an $H$-field with one index along their worldvolume.

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## A. Characters and modular transformations

Here we collect relevant character and modular transformation formulae for easy reference.

### A.1 Level $k$ theta functions

We define the $\theta$ functions as

$$
\Theta_{m,k}(\tau,z) \equiv \sum_{\ell\in\mathbb{Z}} e^{2\pi i\tau k(\ell+\frac{m}{2k})^2} e^{2\pi izk(\ell+\frac{m}{2k})}\,. \tag{A.1}
$$

Under modular transformations they transform as

$$
\Theta_{m,k}(\tau,z) = \frac{1}{\sqrt{\tau/i}} q'^{kz^2/4} \frac{1}{\sqrt{2k}} \sum_{m'} e^{-i\pi mm'/k}\Theta_{m',k}\left(-\frac{1}{\tau},-\frac{z}{\tau}\right), \tag{A.2}
$$

where $q' \equiv e^{-2\pi i/\tau}$

### A.2 SU(2) characters

The SU(2) characters are given by

$$\chi^{\mathrm{su}(2)}(\tau,z) \equiv q^{-c/24}\,\mathrm{Tr}_j[q^{L_0}e^{i2\pi z J_0^3}] \tag{A.3}$$
$$= q^{-c/24}\frac{\sum_m q^{\frac{1}{k+2}[(j+\frac{1}{2}+m(k+2))^2-1/4]}\left(e^{i2\pi z(j+\frac{1}{2}+m(k+2))}-e^{-i2\pi z(j+\frac{1}{2}+m(k+2))}\right)}{(e^{i\pi z}-e^{-i\pi z})\prod(1-e^{i2\pi z}q^n)(1-e^{-i2\pi z}q^n)(1-q^n)}\,.$$

In the above form we can see that the denominators come from the three currents and the numerator takes into account the presence of null states.

This character can also be written as a ratio of theta functions

$$\chi_j(\tau,z) = \frac{\Theta_{(2j+1),k+2}-\Theta_{-(2j+1),k+2}}{\Theta_{1,2}-\Theta_{-1,2}}\,. \tag{A.4}$$

We will use the formula

$$\chi_j^{\mathrm{su}(2)}(\tau,z) = e^{-i2\pi k z^2/(4\tau)}S_j^{\ j'}\chi_{j'}^{\mathrm{su}(2)}\left(-\frac{1}{\tau},-\frac{z}{\tau}\right)\,. \tag{A.5}$$

# B. Computation of overlaps between A- and B-branes in the parafermion theory

We begin the computation by computing overlaps of A-type and B-type Ishibashi states. We act on (3.17) with $q_c^{H^{\mathrm{SU}(2)}}$ where $H = L_0 - c/24$ and take an inner product with an A-type Ishibashi state. Using

$$\langle\langle A,n,n|q_c^{H^{\mathrm{U}(1)}}|B,r,-r\rangle\rangle = \delta_{n,0}\delta_{r,0}\chi_{ND}(q_c)\,, \tag{B.1}$$

where

$$\chi_{ND}(q_c) = \frac{1}{q_c^{1/24}\prod(1+q_c^n)} \tag{B.2}$$

we obtain

$$\langle\langle Aj0|q_c^{L_0-\frac{c}{24}}|Bj'0\rangle\rangle = \delta_{j,j'}\chi'_j\,. \tag{B.3}$$

Here $\chi'_j$ is defined in terms of SU(2) characters as

$$\chi'_j = \frac{\chi_j^{\mathrm{su}(2)}(\tau,z=1/2)}{\chi_{ND}} \tag{B.4}$$

and we are thinking of the SU(2) Ishibashi state as a product of a parafermion state and a U(1) state. Here $\chi^{\mathrm{su}(2)}(\tau,z)$ is the SU(2) character given in (A.3). (Notice that $\chi_j(\tau,z=1/2)$ vanishes for half integer $j$.)

We will need to use the modular transformation properties of these characters. Let us start with the modular transformation property of the single boson character with ND boundary conditions. It has a simple form

$$\chi_{ND}(q_c) = \frac{1}{q_c^{1/24}\prod_{n=1}^{\infty}(1+q_c^n)} = \frac{\eta(\tau_c)}{\eta(2\tau_c)} = \sqrt{2}\frac{\eta(\tau_o)}{\eta(\frac{1}{2}\tau_o)} = \sqrt{2}\frac{q_o^{1/48}}{\prod_{n=1}^{\infty}\left(1-q_o^{n-\frac{1}{2}}\right)} \tag{B.5}$$

here $q_c = e^{2\pi i\tau_c}$ is the closed string modular parameter and $q_o = e^{-2\pi i/\tau_c} = e^{2\pi i\tau_o}$ the open string one. The formula (B.5) can be understood simply by thinking of $2/\chi_{ND}^2(q_c)$ as the partition function of two fermions with $+,-$ boundary conditions on a torus.

Now we transform the SU(2) characters using (A.5). Setting $z = 1/2$ we find, for the parafermion characters we are interested in

$$\chi_j'(q_c) = \frac{1}{\sqrt{2}}S_{j,j'}\widetilde{\chi}_{j'}(q_o)\,, \tag{B.6}$$

where $\widetilde{\chi}_j$ is defined as

$$\widetilde{\chi}_j(q_o) = q_o^{k/16}\chi_j^{\mathrm{SU}(2)}\left(\tau',\frac{\tau'}{2}\right)q_o^{1/24}q_o^{-1/16}\prod_{n=1}^{\infty}\left(1-q_o^{n-\frac{1}{2}}\right). \tag{B.7}$$

Notice that $\widetilde{\chi}_j = \widetilde{\chi}_{\frac{k}{2}-j}$. This can be seen most easily from the the modular transformation property and the fact that $\chi_j'$ vanishes for half integer $j$.

Finally we can compute

$$\begin{aligned}{}_C\langle B\widehat{j}'|q_c^{L_0-\frac{c}{24}}|A\widehat{j}\widehat{m}\rangle_C &= (2k)^{1/4}\sum_{j\in\mathbb{Z}}\frac{S_{\widehat{j}'}{}^{j}\,S^{PF}{}_{\widehat{j}\widehat{m}}{}^{j0}}{\sqrt{S_0{}^{j}}\sqrt{S^{PF}{}_{00}{}^{j0}}}\chi'(q_c) = \sum_{j\in\mathbb{Z}}\frac{S_{\widehat{j}',j}S_{\widehat{j}j}S_{j,j''}}{S_{0j}}\widetilde{\chi}_{j''}(q_c)\\ &= \sum_{j''}N_{\widehat{j}',\widehat{j}}^{j''}\widetilde{\chi}_{j''}(q_o)\,,\end{aligned} \tag{B.8}$$

where in the second line we used the relation between the parafermion S-matrix and the SU(2) S-matrix and the modular transformation properties of $\chi_j'$. We see that indeed the coefficients are all integers.

In order to compute the overlaps with the special B-branes with $\widehat{j} = k/4$ (when $k$ is even) we need the overlap

$$\left\langle\left\langle A\frac{k}{4}\frac{k}{2}\right| q_c^{L_0-\frac{c}{24}}\left| B\frac{k}{4}\frac{k}{2}-\frac{k}{2}\right\rangle\right\rangle. \tag{B.9}$$

This overlap can be computed in terms of the S-matrix for the 1-point blocks on the torus.

Referring to the decomposition (3.17) we see that the parafermionic state, which exists in the closed string spectrum, multiplies a state in the U(1) theory which has squareroot branch cuts with representations with $r$ odd. We can cure this problem by acting with the operator in the SU(2) theory:

$$\Phi^{j=k/2}_{m=k/2}(z) = \Phi^{k/2}_{k}(z) \otimes e^{i\frac{k}{\sqrt{2k}}\phi}(z) = 1 \otimes e^{i\frac{k}{\sqrt{2k}}\phi}(z)\,. \tag{B.10}$$

We then obtain

$$\langle\langle j'|(1\otimes\widetilde{\Phi}^{k/2}_{k/2}(\bar{z}))q_c^{H^{\mathrm{SU}(2)}}(1\otimes e^{i\pi\widetilde{J}^1_0})|j\rangle\rangle = \chi_{ND}\left\langle\left\langle A\frac{k}{4}\frac{k}{2}\right| q_c^{L_0-\frac{c}{24}}\left|B\frac{k}{4}\frac{k}{2}-\frac{k}{2}\right\rangle\right\rangle. \tag{B.11}$$

The left-hand side is a conformal block for the one-point function on the torus of the $j = k/2$ operator ("simple current"). Analogs of the Verlinde formula for these blocks were worked out in [30, 31] and have been applied to boundary conformal field theory in [6, 34]. In this way one can complete the computation of the overlap, but we have not carried out the details.

Finally, let us consider the B-A type overlaps in the superparafermionic theory. From the coset decomposition we obtain the equality of Ishibashi states

$$|j\rangle\rangle \otimes |s\rangle\rangle = \sum_{r=0}^{2k+3} |j, r, s\rangle\rangle \otimes |r\rangle\rangle\,. \tag{B.12}$$

The B-type states are now defined by

$$(1 \otimes e^{i\pi\widetilde{J}^1_0})|j\rangle\rangle \otimes |Bs, -s\rangle\rangle = \sum_{r=0}^{2k+3} |B, j, r, s; -r-s\rangle\rangle \otimes |B, r, -r\rangle\rangle \tag{B.13}$$

from which we obtain

$$\langle\langle Aj, n, s|q_c^{L_0-\frac{c}{24}}|Bj', n', s'\rangle\rangle = \delta_{n,0}\delta_{n',0}\delta_{s,0}\delta_{s',0}\delta_{j,j'}\chi^{\mathrm{SU}(2)}_j\left(\tau_c, z=\frac{1}{2}\right) \tag{B.14}$$

for $j \neq k/4$, with more elaborate formulae in this latter case. In (B.14) the delta functions should be understood up to the spectral flow identification. The twisted-scalar function $\chi_{ND}$ has cancelled from the LHS and RHS. From this expression one easily checks the integrality of the coefficients in the open string channel for A-B overlaps.

## C. Explicit low $k$ examples of branes in bosonic parafermions

The purpose of this appendix is to demonstrate our general formalism in special solvable examples at low values of $k$.

### C.1 $k = 2$

The simplest parafermion theory is

$$\frac{\mathrm{SU}(2)_2}{\mathrm{U}(1)_2}\,, \tag{C.1}$$

which is the same as the Ising model. Its fields are easily identified as

$$\begin{array}{ccc} (j,m) & \Delta & \\ (0,0) & 0 & \mathbf{1} \\ (1,0) & \frac{1}{2} & \psi \\ (\frac{1}{2},1) & \frac{1}{16} & \sigma \end{array} \tag{C.2}$$

The three Cardy states $|A,0,0\rangle_C$, $|A,1,0\rangle_C$ and $|A,1/2,1\rangle_C$ have masses proportional to 1, 1 and $\sqrt{2}$, respectively. At small level one cannot necessarily trust the geometrical picture. Nevertheless, it works remarkably well. Geometrically, we can interpret the first two states as D0-branes at the two special points on the disk and the third is a D1-brane connecting them. The spectrum of open string on these D-branes is $\chi_{0,0}$, $\chi_{0,0}$ and $\chi_{0,0}+\chi_{1,0}$, respectively.

Our general discussion predicts three B-branes at the center of the disk $|B,\widehat{j}=0\rangle_C$, $|B,\widehat{j}=1/2,\eta=\pm 1\rangle_C$, which are made out of a single or two D0-branes. Their masses should be $\sqrt{2}$, 1 and 1, respectively and with spectrum of open strings on them $\chi_{0,0}+\chi_{1,0}$, $\chi_{0,0}$ and $\chi_{0,0}$. Since this is a discrete series theory, it should not have any more Cardy states in addition to the three A-branes mentioned above. Indeed, it is easy to identify $|B,\widehat{j}=0\rangle_C=|A,1/2,1\rangle_C$ and $|B,\widehat{j}=1/2,\eta=\pm 1\rangle_C=|A,\widehat{j}=0,\widehat{m}=0,1\rangle_C$.

The fact that in this model the A-branes are the same as the B-branes is a consequence of the self-duality of the Ising model.

### C.2 $k=3$

The second simplest parafermion theory is

$$\frac{\mathrm{SU}(2)_3}{\mathrm{U}(1)_3}\,. \tag{C.3}$$

It has $c=4/5$ and representations:

$$\begin{array}{cc} (j,n) & \Delta \\ (0,0)\sim(\frac{3}{2},3)\sim(\frac{3}{2},-3) & 0 \\ (1,0)\sim(\frac{1}{2},3)\sim(\frac{1}{2},-3) & \frac{2}{5} \\ (\frac{1}{2},\pm 1)\sim(1,\pm 2) & \frac{1}{15} \\ (\frac{3}{2},\pm 1)\sim(0,\pm 2) & \frac{2}{3} \end{array} \tag{C.4}$$

Since $c$ is less than one, this theory is also a member of the Virasoro discrete series ($p=4$ when $c=1-\frac{6}{p(p+1)}$), and hence can be represented as the coset $\frac{\mathrm{SU}(2)_3\times \mathrm{SU}(2)_1}{\mathrm{SU}(2)_4}$. The parafermion theory is not the standard modular invariant of this theory but the one with an extended chiral algebra. It can be denoted as $\frac{\mathrm{SU}(2)_3\times\mathrm{SU}(2)_1}{\mathrm{SO}(3)_4}$. The representations of the discrete series are labeled by two integers: $r=2j_1+1=1,\ldots,4$ and $s=2j_2+1=1,\ldots,5$ corresponding to the spins of two of the SU(2) factors in the coset (the spin of $\mathrm{SU}(2)_1$ is determined by a selection rule) and they are subject to the identification $(r,s) \sim (5-r, 6-s)$. The dimensions of the representations are

$$\Delta_{rs} = \frac{(6r-5s)^2-1}{120} \tag{C.5}$$

leading to the spectrum of dimensions

$$\begin{array}{ccccc} sr & 1 & 2 & 3 & 4 \\ 1 & 0 & \frac{2}{5} & \frac{7}{5} & 3 \\ 2 & \frac{1}{8} & \frac{1}{40} & \frac{21}{40} & \frac{13}{8} \\ 3 & \frac{2}{3} & \frac{1}{15} & \frac{1}{15} & \frac{2}{3} \\ 4 & \frac{13}{8} & \frac{21}{40} & \frac{1}{40} & \frac{1}{8} \\ 5 & 3 & \frac{7}{5} & \frac{2}{5} & 0 \end{array} \tag{C.6}$$

In terms of these representations the partition function of the parafermion theory is

$$Z_{PF} = |\chi_{11} + \chi_{15}|^2 + |\chi_{21} + \chi_{25}|^2 + 2|\chi_{13}|^2 + 2|\chi_{23}|^2 \,. \tag{C.7}$$

Note that as expected from the coset with $\mathrm{SO}(3)_4$ as opposed to $\mathrm{SU}(2)_4$, the chiral algebra was extended (by $\chi_{15}$) and correspondingly, all representations with $s$ even (half integer spin) were projected out. Also, the two representations with characters $\chi_{13}$ which are isomorphic as Virasoro representations are different representations of the parafermion algebra (similarly for $\chi_{23}$). We will distinguish these representations by denoting them by $13\pm$ and $23\pm$. These representations can be identified with the parafermion representations as

$$\begin{aligned} \chi_{11} + \chi_{15} &= \chi_{j=0,n=0} \,, \\ \chi_{13\pm} &= \chi_{j=0,n=\pm 2} \,, \\ \chi_{21} + \chi_{25} &= \chi_{j=1,n=0} \,, \\ \chi_{23\pm} &= \chi_{j=\frac{1}{2},n=\pm 1} \,. \end{aligned} \tag{C.8}$$

The Ishibashi states in the discrete series theory with the standard modular invariant $\sum_{rs} |\chi_{rs}|^2$ are determined by invariance under the Virasoro algebra. They are $|rs\rangle\rangle$ for all $r, s$ subject to the identification.

Let us examine the Ishibashi states in the parafermion theory. Imposing invariance under the whole parafermion chiral algebra we find

$$\begin{aligned} |11\rangle\rangle + |15\rangle\rangle &= |A, \widehat{j}=0, \widehat{n}=0\rangle\rangle \,, \qquad & |13\pm\rangle\rangle &= |A, \widehat{j}=0, \widehat{n}=\pm 2\rangle\rangle \,, \\ |21\rangle\rangle + |25\rangle\rangle &= |A, \widehat{j}=1, \widehat{n}=0\rangle\rangle \,, \qquad & |23\pm\rangle\rangle &= |A, \widehat{j}=\frac{1}{2}, \widehat{n}=\pm 1\rangle\rangle \,. \end{aligned} \tag{C.9}$$

Out of these states we can form Cardy states

$$\begin{aligned} &|A, \widehat{j}=0, \widehat{n}=0\rangle_C \,, \qquad && |A, \widehat{j}=0, \widehat{n}=\pm 2\rangle_C \,, \\ &|A, \widehat{j}=1, \widehat{n}=0\rangle_C \,, \qquad && |A, \widehat{j}=\frac{1}{2}, \widehat{n}=\pm 1\rangle_C \end{aligned} \tag{C.10}$$

by using

$$
\begin{aligned}
|A,\hat{j},\hat{n}\rangle_C &= \sum_{j'n'} \frac{S^{PF}_{\hat{j}\hat{n}j'n'}}{\sqrt{S^{PF}_{00j'n'}}} |A,j'n'\rangle\rangle\,, \\
S^{PF}_{\hat{j}\hat{n}j'n'} &= \frac{2}{\sqrt{15}} e^{\frac{i\pi\hat{n}n'}{3}} \sin\frac{\pi(2\hat{j}+1)(2j'+1)}{5}\,.
\end{aligned} \tag{C.11}
$$

The two Ishibashi states

$$
|11\rangle\rangle - |15\rangle\rangle\,, \qquad |21\rangle\rangle - |25\rangle\rangle \tag{C.12}
$$

satisfy all the consistency conditions including those arising from the Virasoro algebra. They are not invariant under the parafermion chiral algebra but this is not a fundamental consistency condition. It is clear that the eight-dimensional space spanned by (C.9) and (C.12) is the entire set of Ishibashi states in the parafermion theory. With these Ishibashi states we identify two new Cardy states

$$
\begin{aligned}
|B,\hat{j}=0\rangle_C &= \frac{\sqrt{2}S_{1211}}{\sqrt{S_{1111}}}(|11\rangle\rangle - |15\rangle\rangle) + \frac{\sqrt{2}S_{1221}}{\sqrt{S_{1121}}}(|21\rangle\rangle - |25\rangle\rangle)\,, \\
|B,\hat{j}=1\rangle_C &= \frac{\sqrt{2}S_{3211}}{\sqrt{S_{1111}}}(|11\rangle\rangle - |15\rangle\rangle) + \frac{\sqrt{2}S_{3221}}{\sqrt{S_{1121}}}(|21\rangle\rangle - |25\rangle\rangle)\,,
\end{aligned} \tag{C.13}
$$

where we have used the $S$ matrix of the minimal model

$$
S_{rsr's'} = \frac{2}{\sqrt{15}}(-1)^{(r+s)(r'+s')} \sin\left(\frac{\pi r r'}{5}\right) \sin\left(\frac{\pi s s'}{6}\right). \tag{C.14}
$$

Using this $S$ matrix we can find the open string spectrum between the various D-branes by calculating the inner products

$$
\begin{aligned}
{}_C\langle B,\hat{j}=0|q_c^{L_0-\frac{c}{24}}|B,\hat{j}=0\rangle_C &= \chi_{11}+\chi_{15}+2\chi_{13} \\
&= \chi_{j=0,n=0}+\chi_{j=0,n=2}+\chi_{j=0,n=-2}\,, \\
{}_C\langle B,\hat{j}=1|q_c^{L_0-\frac{c}{24}}|B,\hat{j}=1\rangle_C &= \chi_{11}+\chi_{15}+2\chi_{13}+\chi_{21}+\chi_{25}+2\chi_{23} \\
&= \chi_{j=0,n=0}+\chi_{j=0,n=2}+\chi_{j=0,n=-2}+\chi_{j=1,n=0}+ \\
&\quad +\chi_{j=\frac{1}{2},n=1}+\chi_{j=\frac{1}{2},n=-1}\,, \\
{}_C\langle B,\hat{j}=0|q_c^{L_0-\frac{c}{24}}|B,\hat{j}=1\rangle_C &= \chi_{21}+\chi_{25}+2\chi_{23} \\
&= \chi_{j=1,n=0}+\chi_{j=\frac{1}{2},n=1}+\chi_{j=\frac{1}{2},n=-1}\,, \\
{}_C\langle B,\hat{j}=0|q_c^{L_0-\frac{c}{24}}|A,\hat{j}=0,\hat{n}\rangle_C &= \chi_{12}+\chi_{14}\,, \\
{}_C\langle B,\hat{j}=0|q_c^{L_0-\frac{c}{24}}|A,\hat{j}=\frac{1}{2},\hat{n}\rangle_C &= \chi_{22}+\chi_{24}\,, \\
{}_C\langle B,\hat{j}=1|q_c^{L_0-\frac{c}{24}}|A,\hat{j}=0,\hat{n}\rangle &= \chi_{22}+\chi_{24}\,, \\
{}_C\langle B,\hat{j}=1|q_c^{L_0-\frac{c}{24}}|A,\hat{j}=\frac{1}{2},\hat{n}\rangle_C &= \chi_{12}+\chi_{14}+\chi_{22}+\chi_{24}\,.
\end{aligned} \tag{C.15}
$$

Here we interpreted the factors of two in the fusion rules to lead to an answer which is compatible with the global symmetries of the parafermion theory.

Note that the characters described in appendix B, which appear when we compute the overlaps between A and B states, are, in this case,

$$\begin{aligned}
\chi'_0 &= \chi_0 - \chi_3\,,\\
\chi'_1 &= -\chi_{2/5} + \chi_{7/5}\,,\\
\widetilde{\chi}_0 &= \widetilde{\chi}_{3/2} = \chi_{1/8} + \chi_{13/8}\,,\\
\widetilde{\chi}_{1/2} &= \widetilde{\chi}_1 = \chi_{1/40} + \chi_{21/40}\,.
\end{aligned} \tag{C.16}$$

It is also worth remarking that the above boundary states are the same as those found by [34] by associating non-trivial representations of minimal model fusion rule algebras to the non-diagonal modular invariants of the $c < 1$ series. They are also mentioned in [6]. The significance of this example is that since this model is a discrete series model, here we know that there are no other D-branes in the system. This will not be the case in the next example.

### C.3 $k = 4$

The next parafermion theory is based on the coset

$$\frac{\mathrm{SU}(2)_4}{\mathrm{U}(1)_4} \tag{C.17}$$

and has $c = 1$. This theory is isomorphic to a boson $\phi$ on $S^1/\mathbb{Z}_2$ with $R^2 = 6$. The spectrum of primary fields is

$$\begin{array}{ccc}
(j,m) & \mathcal{O} & \Delta\\
(0,0) & \mathbf{1} & 0\\
(\frac{1}{2},\pm1) & \sigma_{1,2} & \frac{1}{16}\\
(1,0) & e^{i2\phi/\sqrt{6}} + e^{-i2\phi/\sqrt{6}} & \frac{1}{3}\\
(1,2) & e^{i\phi/\sqrt{6}} + e^{-i\phi/\sqrt{6}} & \frac{1}{12}\\
(\frac{1}{2},\pm3) & \tau_{1,2} & \frac{9}{16}\\
(0,4) & \partial\phi & 1\\
(0,\pm2) & e^{i3\phi/\sqrt{6}} \pm e^{-i3\phi/\sqrt{6}} & \frac{3}{4}
\end{array} \tag{C.18}$$

where $\mathcal{O}$ is the operator in terms of the orbifold theory. Here the chiral algebra $\mathcal{A}$ includes the even (under $\phi \to -\phi$) operators in the chiral algebra of the rational torus; e.g. $e^{i\sqrt{6}\phi} + e^{-i\sqrt{6}\phi}$. The fields $\sigma_{1,2}$ and $\tau_{1,2}$ are twist fields located at the singularities of the orbifold.

We easily construct 10 Ishibashi $|A, jm\rangle\rangle$ states which are invariant under $\mathcal{A}$ and combine them into 10 Cardy states $|A, \widehat{j}\widehat{m}\rangle_C$. These 10 Cardy states can be viewed either as states in the orbifold $S^1/\mathbb{Z}_2$ theory or in the disk theory. The four states $|A, \widehat{j} = 0, \widehat{m} = 0, \pm2, 4\rangle_C$ correspond to D0-branes at the four special points on the boundary of the disk. The four states $|A, \widehat{j} = 1/2, \widehat{m} = \pm1, \pm3\rangle_C$ correspond to D1-branes stretched between adjacent points on the boundary. The two states $|A, \widehat{j} = 1, \widehat{m} = 0, 2\rangle_C$ correspond to D1-branes stretched between points on the boundary which are not adjacent.

In terms of the orbifold theory these 10 states can be interpreted as follows. The four states $|A, \widehat{j} = 0, \widehat{m} = 0, \pm 2, 4\rangle_C$ are D0-branes stuck at the orbifold singularities. The four states $|A, \widehat{j} = 1/2, \widehat{m} = \pm 1, \pm 3\rangle_C$ are D1-branes wrapping the orbifold once with the four allowed values of the Wilson lines. The two other states $|A, \widehat{j} = 1, \widehat{m} = 0, 2\rangle_C$ are two D0-brane states at points along the interval $S^1/\mathbb{Z}_2$ corresponding to angles $\pm\pi/3$ and $\pm 2\pi/3$ in the $S^1$ (the $\mathbb{Z}_2$ identification is $\alpha \sim -\alpha$). This interpretation is consistent with the fact that the masses of these D-branes are proportional to $1/2$, $\sqrt{3}/2$ and $1$, respectively, and with the spectrum of open strings stretched between them.

It is clear from the orbifold point of view that the two D0-branes on the interval can move and are part of a moduli space of D0-branes. Indeed, the open strings living on these D-branes include massless moduli. It is not clear what this moduli corresponds to in the disk picture.

From the general parafermion analysis we expect also four more D-brane states at the center of the disk $|B, \widehat{j} = 0, 1/2\rangle_C$, $|B, \widehat{j} = 1, \eta = \pm 1\rangle_C$. In the units we used above their masses are $1$, $\sqrt{3}$, $1$ and $1$. We identify them in the $S^1/\mathbb{Z}_2$ picture as follows. $|B, \widehat{j} = 0\rangle_C$ is a D0-brane at the center of the orbifold (angle $\pm\pi/2$ on the covering $S^1$), $|B, \widehat{j} = 1/2\rangle_C$ is two D1-branes wrapping the orbifold with opposite values of the Wilson line (recall that in this configuration the Wilson line is a modulus) at the symmetric value of the Wilson line, and $|B, \widehat{j} = 1, \eta = 1\rangle_C$ ($|B, \widehat{j} = 1, \eta = -1\rangle_C$) is a D0-brane at $\pm\pi/6$ ($\pm 5\pi/6$). This interpretation is supported by their masses, the spectrum of open strings on them, and the spectrum of open strings stretched between them. The massless open string on $|B, j = 1/2\rangle_C$ is interpreted in the orbifold picture as the arbitrary value of the Wilson line, and the massless open string on $|B, j = 1, \eta\rangle_C$ is interpreted as the position of the D0-brane.

This example is special because some of the branes we found are part of a continuum of D-branes. The generic D-brane on the moduli space is not invariant under such a large chiral algebra and hence it is neither an A-brane nor a B-brane. The first signal of such a moduli space is the existence of massless open strings on the branes. Such massless states ($\Delta = 1$) occur also for larger values of $k$, but it is easy to see that the three point function of such open string vertex operators are non zero. Therefore, they are not moduli and the D-branes are isolated.

# D. The shape of the branes

## D.1 Preliminaries

In this appendix we will determine the shape of the branes. We will always consider large $k$, since only in that case we can talk about the shape of the brane in a classical geometrical way. We first do the calculation for SU(2) as in [43] and then we proceed with the other cases. What we will do is to scatter a massless closed string from the brane.[14]

This scattering amplitude is determined in terms of the overlap

$$\langle \text{Boundary state} \text{ — } \text{closed string} \rangle \,. \tag{D.1}$$

We can think of the closed string state as a graviton with polarizations in the other directions (directions not involving the WZW model). In that case the wavefunction of the closed string in the WZW directions involves only the primary states and is characterized by $|\text{closed string}\rangle = |j, m, m'; 0\rangle$, where the zero means that no descendant appears. These states can be thought of as functions over the manifold for $j \ll k$.

In this appendix we take $m = 0, 1/2, \ldots$. It is related to $n$ used in the main text through $n = 2m$.

When we scatter a closed string state we can pick any wavefunction for it. A particularly useful function is a delta function on the manifold. For the SU(2) manifold we choose $\delta(\vec{\theta} - \vec{\theta}')$, where $\vec{\theta}$ denotes the three angles in some coordinate system.

Let us define some useful functions. The eigenfunctions of the Laplacian on the group manifold can be written as

$$\mathcal{D}^j_{mm'} = \langle jm|g(\vec{\theta})|jm'\rangle \,, \qquad \langle jm|jm'\rangle = \delta_{m,m'} \,, \tag{D.2}$$

where $|j, m\rangle$ are a basis for the spin $j$ representation of SU(2). We can determine the relation between the normalization of $\mathcal{D}^j_{mm'}$ and $|j, m, m'\rangle$ (which is normalized to one) in the following way. First notice that $|j, m\rangle$ in (D.2) is normalized to one. Then if we act with raising or lowering operators of $\mathrm{SU}(2)_L$ or $\mathrm{SU}(2)_R$ then we would produce the same factors on $\mathcal{D}$ as they produce on $|j, m, m'\rangle$. Therefore, the relation between the two normalizations is just a constant independent of $m, m'$. This constant can be determined by taking the square of (D.2) and summing over $m'$, so that we get $\sum_{m'} |\langle jm|g(\vec{\theta})|jm'\rangle|^2 = 1$. We find that

$$\mathcal{D}^j_{mm'} \sim \frac{1}{\sqrt{2j+1}} \langle \vec{\theta}|j, m, m'\rangle \,. \tag{D.3}$$

So that we can write a $\delta$ function as

$$\begin{aligned} \delta(\vec{\theta} - \vec{\theta}') &\sim \sum_{j,m,m'} (2j+1) \mathcal{D}^j_{mm'}(\vec{\theta})^* \mathcal{D}^j_{mm'}(\vec{\theta}') \\ &\sim \sum_{j,m,m'} \sqrt{2j+1} \mathcal{D}^j_{mm'}(\vec{\theta})^* \langle \vec{\theta}'|j, m, m'\rangle \,. \end{aligned} \tag{D.4}$$

[14] The computation which follows has some similarities to the computations of [43], but in fact is different. Reference [43] computes the change in the metric and $B$-field induced by the presence of a brane in the WZW model. The computation is analogous to measuring the transverse metric of a flat brane in flat space $h(r) \sim \text{const.}/r^{d_T - 2}$ where $d_T$ is the number of transverse dimensions.

Here the ranges of indices are $|m|, |m'| \leq j$ and $j = 0, 1/2, \ldots$. We only want to consider closed string states which are well localized. Therefore, we will really consider a smeared delta function where states with $j \sim k$ are suppressed. We could do this with a cutoff like $\exp[-j^2/\epsilon^2]$. We will let $k \to \infty$ and then take $\epsilon \to 0$.

### D.2 Shape of A-branes in SU(2)

Now let us consider the boundary states in the SU(2) WZW theory which correspond to the usual conjugacy classes which we denote by $|A, \widehat{j}\rangle$. Defining $\widehat{\psi} \equiv \pi(2\widehat{j}+1)/(k+2)$ we find

$$\begin{aligned} {}_C\langle A, \widehat{j}|\vec{\theta}\rangle &\sim \sum_{jm} \frac{S_{\widehat{j},j}}{\sqrt{S_{0,j}}} \sqrt{2j+1}\mathcal{D}^j_{mm}(\vec{\theta})^* \\ &\sim \frac{2^{1/4}}{\sqrt{\pi}}(k+2)^{1/4} \sum_j \sin[(2j+1)\widehat{\psi}] \frac{\sin(2j+1)\psi}{\sin\psi} \\ &\sim \frac{\sqrt{\pi}}{2^{7/4}} \frac{(k+2)^{1/4}}{\sin\widehat{\psi}} \left(\delta(\widehat{\psi}-\psi) - \delta(\psi - \widehat{\psi} + \pi) + \cdots\right) \end{aligned} \tag{D.5}$$

where we have used approximations valid for $j \ll k$, e.g. so that $S_{0j} \sim (2j+1)/(k+2)^{3/2} \sim (2j+1)/k^{3/2}$. This is justified by the smearing of the delta function as discussed above. In the periodic delta function only the first term contributes because of the restriction on the ranges of $\psi$. We also used that the boundary state has the same $m$ for the left and right movers and the fact that $\sum_m \mathcal{D}^j_{mm} = \frac{\sin(2j+1)\psi}{\sin\psi}$. Note also that $\psi$ is the angle in the coordinates where the metric of the $S^3$ is

$$ds^2 = d\psi^2 + \sin^2\psi d^2 s_{S^2} \tag{D.6}$$

so we can think of a point parametrized by $\psi$ as a rotation in SU(2) by angle $2\psi$.

We can see we get the expected result, saying that branes are localized at the conjugacy class given by $\widehat{\psi}$. The factor of $1/\sin\widehat{\psi}$ is also reasonable, reflecting the fact that the tension of the D2-brane is getting stronger of small $\widehat{\psi}$ since $F$ is getting stronger. This factor in the effective tension gives also the right mass after integration over the $S^2$, $M \sim k\sin\widehat{\psi}$.

Note that in the above calculation there is no restriction on the size of $\widehat{\psi}$, it could be very big and it could be a brane that wraps the equator, or even branes that are close to $\widehat{\psi} \sim \pi$. (Of course if the size of $\widehat{\psi}$ is too small then the approximate $\delta$ function we were considering above would not resolve the brane appropriately).

### D.3 Shape of B-branes in SU(2)

Now let us consider the B-branes defined in (6.4). We will disregard the second term in the large-$k$ limit. When we multiply by the delta function state we find

$$\langle B, \widehat{j}, \eta|\vec{\theta}\rangle \sim k \sum_j \mathcal{D}^j_{00} S_{\widehat{j}j}\,. \tag{D.7}$$

It is convenient to think about $\mathcal{D}^j_{00}$ in Euler angles $\chi, \widetilde{\theta}, \varphi$, see (2.18). Then we need to compute $\langle 0|e^{i\widetilde{\theta}J^1}|0\rangle$. These are the Legendre polynomials $P_j(\cos\widetilde{\theta})$ (note that only integer values of $j$ appear), including the normalizations since $P_n(1) = 1$.

In order to compute the sum in (D.7) we should remember the generating function formula for Legendre polynomials

$$\sum_{n=0} t^n P_n(x) = \frac{1}{\sqrt{1-2tx+t^2}} \,. \tag{D.8}$$

So we see that, with $x = \cos\widetilde{\theta}$,

$$\langle B\widehat{j}|\widetilde{\theta}\rangle \sim -ie^{i\widehat{\psi}} \sum_{n=0}^{\infty} P_n(x) e^{in2\widehat{\psi}} + \text{c.c.} \sim \frac{\Theta(\cos\widetilde{\theta} - \cos 2\widehat{\psi})}{\sqrt{\cos\widetilde{\theta} - \cos 2\widehat{\psi}}} \,, \tag{D.9}$$

where c.c. means complex conjugate and $\Theta(z)$ is the usual step function which vanishes when $z < 0$.

This same calculation can be applied to the parafermion case. One finds a disk of radius $\rho = \sin(\pi(2\widehat{j}+1)/(k+2))$. It can be checked that the $\widetilde{\theta}$ dependent factor in (D.9) is the expected factor resulting from the open string metric and dilaton in (5.7).

### D.4 The shape of branes in the parafermion theory

Let us now carry out an analogous computation in the parafermion theory. Referring to the geometry discussed in section 2.2 we see that wavefunctions on the disk are SU(2) wavefunctions that are invariant under translations of $\widetilde{\phi}$. So the wavefunctions of the parafermion theory are

$$\begin{aligned} \Psi_{j,m} &= e^{i\alpha_{j,m}} \langle j,m|g|j,-m\rangle = e^{i2m\phi}\langle j,m|e^{i(2\theta-\pi)J^1}|j,m\rangle \\ &= \frac{1}{\sqrt{2j+1}} \langle \theta,\phi|j,m,m\rangle_{PF} \,, \end{aligned} \tag{D.10}$$

where the last equality is just a statement of the normalization of the wavefunctions and $\alpha_{jm}$ is a phase that is irrelevant for our purposes. Let us just remark that these functions are not the usual spherical harmonics

$$Y_{l,m}(\widetilde{\theta},\varphi) = \langle l,0|g|l,m\rangle \,, \tag{D.11}$$

but they are related to them. Of course for $m = 0$ they are the same as the Legendre polynomials we used earlier.

We can now build the delta function closed string state roughly as before

$$\delta(\vec{\theta} - \vec{\theta}') \sim \sum_{jm} (2j+1) \Psi(\vec{\theta})^*_{jm} \Psi(\vec{\theta}')_{jm} = \sum_{jm} \sqrt{2j+1}\, \Psi(\vec{\theta})^*_{j,m} \langle \theta'|j,m,-m\rangle \,, \tag{D.12}$$

where $|j,m,-m\rangle$ is a closed string state normalized to one.

The Cardy state is given in (3.5). Now in order to see what state this is we should take the overlap with the delta function state. We get

$$
\begin{aligned}
{}_C\langle A,\widehat{j},\widehat{m}|\theta,\phi\rangle &\sim \sum_j \sum_{m=-j}^{j} S_{\widehat{j},j} e^{\frac{i4\pi m\widehat{m}}{k}} \Psi_{j,m}(\theta,\phi)^* \\
&= \sum_j \sum_{m=-j}^{j} \sin(\widehat{\psi}(2j+1))\Psi_{j,m}\left(\theta,\phi-\frac{2\pi\widehat{m}}{k}\right)^* . \qquad \text{(D.13)}
\end{aligned}
$$

So we see that $\widehat{m}$ is an angular position. Notice that we restricted the sum over $m$ only to those states which can be thought of as wavefunctions on the space. So we can set $\widehat{m}$ =0 and get the other states by acting with $\mathbb{Z}_k$ rotations in $\phi$. So we need to calculate

$$
\sum_{m=-j}^{j} \langle j,m|e^{i2\theta J^1}|j,-m\rangle e^{i2m\phi} = \mathrm{Tr}[e^{i2\phi J^3} e^{i(2\theta-\pi)J^1}] = \frac{\sin(2j+1)\psi}{\sin\psi}\,, \qquad \text{(D.14)}
$$

where

$$
\cos\psi = \cos\phi\sin\theta\,. \qquad \text{(D.15)}
$$

When we insert this back into (D.13) we find

$$
{}_C\langle A,\widehat{j},\widehat{m}=0|\theta,\phi\rangle \sim \frac{\delta(\widehat{\psi}-\psi(\theta,\phi))}{\sin\psi}\,. \qquad \text{(D.16)}
$$

Notice that this delta function defines a line in the two-dimensional manifold where the parafermion theory lives. A straight vertical line in polar coordinates obeys $\rho\cos\phi = \sin\theta\cos\phi =$ constant, which is indeed what we find from (D.14) and (D.16). Straight lines at other angles are obtained by performing rotations, which amounts to taking non-zero $\widehat{m}$. In the super symmetric case we rotate by $2\pi\widehat{m}/(k+2)$, and we must include the sector from the $\mathrm{U}(1)_2$ theory.

## E. Some formulae from $\mathcal{N}=2$ representation theory, minimal models, and all that

The U(1) generator of the $\mathcal{N}=2$ algebra is normalized so that

$$
\begin{aligned}
[J_n,J_m] &= \frac{c}{3} m\delta_{n+m,0}\,, \\
J(z)J(w) &\sim \frac{c/3}{(z-w)^2} + \cdots \qquad \text{(E.1)}
\end{aligned}
$$

and in particular $[J_n,G^{\pm}_m] = \pm G^{\pm}_{n+m}$.

The spectral flow operation by $\eta$ is [44]

$$\begin{aligned}
L_n &\longrightarrow L_n + \eta J_n + \frac{c}{6}\eta^2 \delta_{n,0}\,, \\
J_n &\longrightarrow J_n + \eta \frac{c}{3}\delta_{n,0}\,, \\
G_r^+ &\longrightarrow G_{r+\eta}^+\,, \\
G_r^- &\longrightarrow G_{r-\eta}^-\,.
\end{aligned} \tag{E.2}$$

The unitary representations were classified in [45]. There is a continuous series of unitary representations with $c \geq 3$ and a discrete series with $c = 3k/(k+2)$, $k \in \mathbb{Z}_+$.

The $\mathcal{N} = 2$ discrete series can be realized by the GKO coset model

$$\frac{\mathrm{SU}(2)_k \times \mathrm{U}(1)_2}{\mathrm{U}(1)_{k+2}}\,. \tag{E.3}$$

The representations are labelled by triples $(j, n, s)$ obtained by decomposing $\mathcal{H}_j^{\mathrm{SU}(2)} \otimes \mathcal{H}_s^{\mathrm{U}(1)_2}$ with respect to a $\mathrm{U}(1)_{k+2}$ subalgebra. Thus representations are labelled by $(j, n, s)$ where $j \in \{0, 1/2, 1, \ldots, k/2\}$, $n$ is an integer defined modulo $2k+4$, and $s$ an integer defined modulo 4. There are four relevant U(1) currents in the model (E.3). These are the SU(2) current $J^3$, the $\mathrm{U}(1)_2$ current $J^F$, the $\mathrm{U}(1)_{k+2}$ current $J^{(k+2)}$ and the $\mathcal{N} = 2$ current $J^{\mathcal{N}=2}$. They are related by

$$\begin{aligned}
J^{\mathcal{N}=2} &= -\frac{2}{k+2} J^3 + \frac{k}{2(k+2)} J^F\,, \\
J^{(k+2)} &= 2J^3 + J^F\,.
\end{aligned} \tag{E.4}$$

If we express the four currents in terms of free scalar fields as

$$\begin{aligned}
J^3 &= i\sqrt{\frac{k}{2}}\partial X\,, \\
J^F &= i2\partial\sigma\,, \\
J^{N=2} &= i\sqrt{\frac{k}{k+2}}\partial H\,, \\
J^{(k+2)} &= i\sqrt{2(k+2)}\partial g
\end{aligned} \tag{E.5}$$

then

$$\begin{aligned}
\sigma &= \sqrt{\frac{k}{k+2}} H + \sqrt{\frac{2}{k+2}} g\,, \\
X &= -\sqrt{\frac{2}{k+2}} H + \sqrt{\frac{k}{k+2}} g\,.
\end{aligned} \tag{E.6}$$

Using these we find

$$\Phi_{j,m}^{\mathrm{SU}(2)} e^{is\sigma/2} = \left( \widetilde{\Phi} \exp\left[ i(\frac{s}{2} - \frac{(s+2m)}{(k+2)}) \sqrt{\frac{k+2}{k}} H \right] \right) \exp\left[ i \frac{2m+s}{\sqrt{2(k+2)}} g \right]. \tag{E.7}$$

Here $\Phi_{j,m}^{\mathrm{SU}(2)}$ is any SU(2) with those quantum numbers, and $\widetilde{\Phi}$ is a field in the super-parafermion theory.

From (E.7) we conclude $n = (2m+s) \mod (2k+4)$, and hence the selection rule

$$2j + n + s = 0 \mod 2\,. \tag{E.8}$$

The full set of equivalence relations on the representation labels $(j, n, s)$ is generated by

$$\begin{aligned}
(j,n,s) &\sim (j, n+2k+4, s)\,,\\
(j,n,s) &\sim (j, n, s+4)\,,\\
(j,n,s) &\sim \left(\frac{k}{2} - j, n+k+2, s+2\right).
\end{aligned} \tag{E.9}$$

From (E.7) which one learns the U(1) charge of the representation $(j, n, s)$ is

$$q_{(j,n,s)} = \left[\frac{s}{2} - \frac{n}{k+2}\right] \mod 2 \tag{E.10}$$

while the representations have $L_0$ values

$$h_{(j,n,s)} = \left[\frac{j(j+1)}{k+2} - \frac{n^2}{4(k+2)} + \frac{s^2}{8}\right] \mod \mathbb{Z}\,. \tag{E.11}$$

For the analysis of tachyons in the open string channels we need more precise values for $h$. In the domain $|n-s| \leq 2j$ one may apply the expression in (E.11) provided one minimizes over all values of $s \mod 4$. Explicitly we find:

$$\begin{aligned}
h_{(j,n,s)} &= \frac{j(j+1)}{k+2} - \frac{n^2}{4(k+2)} + \frac{s^2}{8}\,, && -2j \leq (n-s) \leq 2j\\
&= \frac{j(j+1)}{k+2} - \frac{n^2}{4(k+2)} + \frac{s^2}{8} + \frac{(n-s-2j)}{2}\,, && 2j \leq (n-s) \leq 2k-2j
\end{aligned} \tag{E.12}$$

which is valid if we fix a fundamental domain $s = 0, 2, \pm 1$ for $s$. Unfortunately, (E.12) does not cover the a full fundamental domain for $n \mod (2k+4)$. Specifically, the exceptional cases $(j, n, s) = (0,0,2), (0,-1,1), (0,1,-1)$ are not covered. However, by moving out of the fundamental domain of $s$ these can be mapped to $(k/2, k+2, 4), (k/2, k+1, 3), (k/2, -k-1, -3)$, respectively. Then the first expression in (E.12) can be used to find the dimensions $3/2$ and twice $\frac{k}{8(k+2)} + 1 = \frac{c}{24} + 1$. In fact, by relaxing the domain of $s$, it is enough to use only the first expression in (E.12). In doing that we should minimize $h$ with respect to $s$ for fixed value of $s \mod 4$.

There is an important operator of fermion number $(-1)^F$. Choosing a fundamental domain it may be defined to be $(-1)^F = +1$ on representations with $s = 0, 1$ and $(-1)^F = -1$ on representations with $s = 2, 3 \sim -1$. The $\mathcal{N} = 2$ algebra mixes these representations, but the chiral algebra does not. In particular the only non-vanishing contributions to the Witten index $\mathrm{Tr}(-1)^F$ come from states $h_{(j,2j+1,+1)} = h_{(j,-2j-1,-1)} = c/24$.

Under spectral flow by $\eta = +1/2$ the representations transform as $(j, n, s) \rightarrow (j, n+1, s+1)$, as one proves using the U(1) charge (E.10).

The characters for the discrete series were found in [46, 47, 48]

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}_{(j,n,s)}} q^{L_0 - c/24} e^{2\pi i z J_0} &= \chi_{(j,n,s)}(\tau, z) \\ &= \sum_{t=0}^{k-1} \chi^{(k)}_{j,n-s+4t}(\tau) \frac{1}{\eta(\tau)} \Theta_{2n+(4t-s)(k+2),2k(k+2)} \left(\tau, -\frac{z}{2k+4}\right). \end{aligned} \tag{E.13}$$

The modular transformation matrix is

$$S_{(j,n,s)}^{\;(j',n',s')} = \frac{1}{k+2} \sin\left(\pi \frac{(2j+1)(2j'+1)}{k+2}\right) e^{i\pi \frac{nn'}{(k+2)}} e^{-i\pi \frac{ss'}{2}}. \tag{E.14}$$

The fusion rules are:

$$N_{jj'}^{\;\;j''} \delta_{n+n'-n''} \delta_{s+s'-s''} + N_{jj'}^{\;\;\frac{k}{2}-j''} \delta_{n+n'-(n''+k+2)} \delta_{s+s'-(s''+2)}, \tag{E.15}$$

where the delta functions are modulo $2k+4$ and 4, respectively.

The discrete symmetry of the model $G$ can readily be found by considering the coset $\frac{\mathrm{SU}(2)_k \times \mathrm{U}(1)_4}{\mathrm{U}(1)_{2k+4}}$. To describe the symmetry group let begin with the symmetry group of the various factors in the coset $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{2k+4}$. Because of the spectral flow identification only a subgroup of this group acts. Moreover, due to the selection rule $2j + n + s \in 2\mathbb{Z}$ only a quotient of this subgroup acts effectively. The resulting symmetry group $G$ is generated by

$$\begin{aligned} g_1 \cdot \Phi_{(j,n,s)} &= e^{2\pi i \left(\frac{n}{2k+4} - \frac{s}{4}\right)} \Phi_{(j,n,s)}, \\ g_2 \cdot \Phi_{(j,n,s)} &= e^{i\pi s} \Phi_{(j,n,s)}. \end{aligned} \tag{E.16}$$

These generators satisfy the relations $g_1^{2k+4} = g_2^k$ and $g_2^2 = 1$. For $k$ even they generate $G = \mathbb{Z}_{2k+4} \times \mathbb{Z}_2$ and for $k$ odd the symmetry group is $G = \mathbb{Z}_{4k+8}$. In the text we usually restrict attention to the $H = \mathbb{Z}_{k+2} \times \mathbb{Z}_2$ subgroup generated by $g_1^2 g_2$ and $g_2$.

## F. Simplest super-parafermion theory $k = 1$

The simplest non-trivial supersymmetric parafermion theory is based on the coset

$$\frac{\mathrm{SU}(2)_1 \times \mathrm{U}(1)_2}{\mathrm{U}(1)_3} = \mathrm{U}(1)_6. \tag{F.1}$$



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It has $c = 1$ and representations:

$$\begin{array}{ccc} (n,s) & \Delta & q \\ (0,0) & 0 & 0 \\ (\pm 2,0) & \frac{2}{3} & \mp\frac{2}{3} \\ (\pm 2,2) & \frac{1}{6} & \pm\frac{1}{3} \\ (0,2) & \frac{3}{2} & 1 \\ (\pm 1,\pm 1) & \frac{1}{24} & \pm\frac{1}{6} \\ (\pm 1,\mp 1) & \frac{25}{24} & \mp\frac{5}{6} \\ (3,\pm 1) & \frac{3}{8} & \mp\frac{1}{2} \end{array} \tag{F.2}$$

We used the identification $(j, s, n) \sim (\frac{1}{2} - j, s + 2 \mod 4, n + 3 \mod 6)$ to set $j = 0$ in all representations and did not write $j$. $q = -\frac{n}{3} + \frac{s}{2} \mod 2$ is the U(1) charge of the representation.

We denote its chiral algebra by $\mathcal{S}_2$. It is the even fermion number elements in the $\mathcal{N} = 2$ superconformal algebra. The representations with $s = 0 \mod 2$ arise from the NS sector. They are combined to the three NS $\mathcal{N} = 2$ representations with $(\Delta, q) = (0, 0), (1/6, \pm 1/3)$. The representations with $s = 1 \mod 2$ arise from the R sector. They combine into the three R $\mathcal{N} = 2$ representations with $(\Delta, q) = (1/24, \pm 1/6), (3/8, 1/2)$.

Since $c$ is less than 3/2, this theory is also a member of the $N = 1$ discrete series and hence can be represented as the coset $\frac{\mathrm{SU}(2)_2 \times \mathrm{SU}(2)_2}{\mathrm{SU}(2)_4}$ (note that the chiral algebra of this coset, which we denote as $\mathcal{S}_1$, is only the even fermion number elements of the $N = 1$ superconformal algebra). The representations of this coset are labeled by the three spins $(j_1, j_2, j_3)$ subject to the selection rule $j_1 + j_2 + j_3 \in \mathbb{Z}$ and the identification $(j_1, j_2, j_3) \sim (1 - j_1, 1 - j_2, 2 - j_3)$. As always, there are two distinct representations at the fixed point of the identification $(j_1 = 1/2, j_2 = 1/2, j_3 = 1)_\pm$. The complete list of representations of the coset is

$$\begin{array}{cc} (j_1,j_2,j_3) & \Delta \\ (0,0,0) & 0 \\ (0,\frac{1}{2},\frac{1}{2}) & \frac{1}{16} \\ (0,1,1) & \frac{1}{6} \\ (1,\frac{1}{2},\frac{1}{2}) & \frac{9}{16} \\ (0,0,1) & \frac{2}{3} \\ (1,1,0) & 1 \\ (0,1,0) & \frac{3}{2} \\ (1,0,0) & \frac{3}{2} \\ (\frac{1}{2},\frac{1}{2},1) & \frac{1}{24} \\ (\frac{1}{2},\frac{1}{2},1)' & \frac{25}{24} \\ (\frac{1}{2},0,\frac{1}{2}) & \frac{1}{16} \\ (\frac{1}{2},\frac{1}{2},0) & \frac{3}{8} \\ (\frac{1}{2},1,\frac{1}{2}) & \frac{9}{16} \end{array} \tag{F.3}$$

The representations with $j_1 \in \mathbb{Z}$ are in the NS sector and those with $j_1 \in \mathbb{Z} + \frac{1}{2}$ are in the R sector. The representations of the $N = 1$ superconformal algebra are

$$\begin{array}{ccc}
(j_1, j_2, j_3) & \Delta & (r,s) \\
(0,0,0) \oplus (1,0,0) & 0 & (1,1) \\
(0,\frac{1}{2},\frac{1}{2}) \oplus (1,\frac{1}{2},\frac{1}{2}) & \frac{1}{16} & (2,2) \\
(0,1,1) \oplus (0,0,1) & \frac{1}{6} & (1,3) \\
(1,1,0) \oplus (0,1,0) & 1 & (1,5) \\
(\frac{1}{2},\frac{1}{2},1) \oplus (\frac{1}{2},\frac{1}{2},1)' & \frac{1}{24} & (2,3) \\
2(\frac{1}{2},0,\frac{1}{2}) & \frac{1}{16} & (1,2) \\
2(\frac{1}{2},\frac{1}{2},0) & \frac{3}{8} & (2,1) \\
2(\frac{1}{2},1,\frac{1}{2}) & \frac{9}{16} & (1,3)
\end{array} \tag{F.4}$$

In the last column we included the $(r, s)$ values of the representation in the Kac table.

The super-parafermion theory is not the standard modular invariant of this theory but the one with an extended chiral algebra. It can be denoted as $\frac{\mathrm{SU}(2)_2 \times \mathrm{SU}(2)_2}{\mathrm{SO}(3)_4}$. Its representations are obtained from the original coset by imposing the selection rule $j_3 \in \mathbb{Z}$, the identification $(j_1, j_2, j_3) \sim (j_1, j_2, 2 - j_3)$ and doubling the representations at the fixed points $(j_1, j_2, j_3 = 1)$ and $(1/2, 1/2, 0)$. This leads to the same spectrum as (F.2)

$$\begin{array}{cccc}
(n,s) & (j_1,j_2,j_3) & \Delta & q \\
(0,0) & (0,0,0) \oplus (1,1,0) & 0 & 0 \\
(\pm 2, 0) & (0,0,1) & \frac{2}{3} & \mp\frac{2}{3} \\
(\pm 2, 2) & (0,1,1) & \frac{1}{6} & \pm\frac{1}{3} \\
(0,2) & (0,1,0) \oplus (1,0,0) & \frac{3}{2} & 1 \\
(\pm 1, \pm 1) & (\frac{1}{2},\frac{1}{2},1)_+ & \frac{1}{24} & \pm\frac{1}{6} \\
(\pm 1, \mp 1) & (\frac{1}{2},\frac{1}{2},1)_- & \frac{25}{24} & \mp\frac{5}{6} \\
(3, \pm 1) & (\frac{1}{2},\frac{1}{2},0) & \frac{3}{8} & \mp\frac{1}{2}
\end{array} \tag{F.5}$$

The partition function of the super-parafermion theory is

$$\begin{aligned} Z_{PF} = {} & |\chi_0 + \chi_1|^2 + 2|\chi_{1/6}|^2 + 2|\chi_{2/3}|^2 + |\chi_{3/2} + \chi_{3/2'}|^2 + 2|\chi_{1/24}|^2 + 2|\chi_{25/24}|^2 + \\ & + 2|\chi_{3/8}|^2 \,, \end{aligned} \tag{F.6}$$

where we have labeled the $\mathcal{S}_1$ characters by the dimension of the primary. The character $\chi_1$ which extends the chiral algebra includes the U(1) current. The two dimension 3/2 fields which are primary fields of this algebra are the two supercharges of the $\mathcal{N} = 2$ algebra. We also recognize the two R ground states with U(1) charges $\pm 1/6$.

This theory clearly has 12 Ishibashi states $|Ans\rangle\rangle$ for $ns$ in (F.2), which are invariant under the full chiral algebra of the theory $\mathcal{S}_2$. Using these states and the S-matrix

$$S_{ns}{}^{n's'} = \frac{1}{\sqrt{12}} e^{\frac{i\pi n n'}{3} - \frac{i\pi s s'}{2}} \tag{F.7}$$

we can construct 12 Cardy states

$$|A,\widehat{n}\widehat{s}\rangle_C = \sum_{\widehat{n}'\widehat{s}'} \frac{S_{\widehat{n}\widehat{s}}{}^{n's'}}{\sqrt{S_{00}{}^{n's'}}}|A,n's'\rangle\rangle. \tag{F.8}$$

(These are exactly the 12 A Cardy states of the $\mathrm{U}(1)_6$ theory.)

Using our standard picture we interpret the target space as a disk with 6 marked points on its boundary. The 12 Cardy states correspond to 12 D-brane states. These are oriented D1-branes stretched between the even or odd marked points.

There are also other branes which respect only $\mathcal{N}=1$ supersymmetry. These are the B-branes which turn out to preserve another $\mathcal{N}=2$ supersymmetry. We start by looking for Ishibashi states preserving $\mathcal{S}_1$. Since the boundary state has to create closed strings which exist in the spectrum, we look at (F.5) and (F.6) and consider the Ishibashi states $|0,0,0\rangle\rangle - |0,0,2\rangle\rangle$ and $|0,1,0)\rangle\rangle - |1,0,0\rangle\rangle$ (we use the notation in terms of $(j_1,j_2,j_3)$ because these are not Ishibashi states of $\mathcal{S}_2$ and therefore cannot be labeled by $ns$). We define the two Cardy states labeled by $\widehat{s}=0,1$

$$|B,\widehat{s}\rangle_C = 3^{1/4}\left[|0,0,0\rangle\rangle - |0,0,2\rangle\rangle + (-1)^{\widehat{s}}(|1,0,0\rangle\rangle - |0,1,0\rangle\rangle)\right]. \tag{F.9}$$

(These are exactly the two B Cardy states of the $\mathrm{U}(1)_6$ theory.) These two states are even and odd Cardy states of the $\mathcal{S}_1$ algebra. They are annihilated by $G(\sigma) - i(-1)^{\widehat{s}}\overline{G}(\sigma)$. We interpret them as a single D0-brane at the center of our disk or a D1-brane around the $\mathrm{U}(1)_2$ circle.

It is easy to compute the inner product

$$\begin{aligned} {}_C\langle B,\widehat{s}|q_c^{L_0-c/24}|B,\widehat{s}'\rangle_C &= \\ &= \sqrt{3}(\chi_{(0,0,0)}(q_c)+\chi_{(0,0,2)}(q_c)+(-1)^{\widehat{s}+\widehat{s}'}(\chi_{(0,1,0)}(q_c)+\chi_{(1,0,0)}(q_c)) \qquad \text{(F.10)} \\ &= \sqrt{3}(\chi_{(0,0)}(q_c)+(-1)^{\widehat{s}+\widehat{s}'}\chi_{(0,2)}(q_c)) = \sum_{ms''}\frac{1+(-1)^{\widehat{s}+\widehat{s}'+s''}}{2}\chi_{(m,s'')}(q_o)\,. \end{aligned}$$

Here we first expressed the answer in terms of $\mathcal{S}_1$ characters, then in terms of $\mathcal{S}_2$ characters, and finally in terms of characters in the open string channel with modular parameter $q_o$.

We see that the open strings living on these two D-branes are the same and include all the NS representations of $\mathcal{S}_2$. The spectrum of string stretched between them is the R representations of $\mathcal{S}_2$.

Using the S-matrix of the $N=1$ models it is easy to compute the inner products

$$\begin{aligned} {}_C\langle B,\widehat{s}'|q_c^{L_0-c/24}|A,\widehat{m},\widehat{s}\rangle_C &= \\ &= \frac{1}{\sqrt{2}}(\chi_{(0,0,0)}(q_c)-\chi_{(0,0,2)}(q_c)+(-1)^{\widehat{s}+\widehat{s}'}(\chi_{(0,1,0)}(q_c)-\chi_{(1,0,0)}(q_c))) \\ &= \frac{1+(-1)^{\widehat{s}+\widehat{s}'}}{2}(\chi_{(0,1/2,1/2)}(q_o)+\chi_{(1,1/2,1/2)}(q_o)) + \\ &\quad + \frac{1-(-1)^{\widehat{s}+\widehat{s}'}}{2}(\chi_{(1/2,0,1/2)}(q_o)+\chi_{(1/2,1,1/2)}(q_o))\,, \end{aligned} \tag{F.11}$$

where all these characters are $\mathcal{S}_1$ characters which are labeled by $(j_1, j_2, j_3)$. We see that we get only $\mathcal{N} = 1$ representations which are not $\mathcal{N} = 2$ representations. Also, for $\widehat{s} = \widehat{s}'$ we get only the representations in the NS sector and for $\widehat{s} \neq \widehat{s}'$ only the representations in the R sector.

To summarize, both the A-branes and the B-branes preserve $\mathcal{N} = 2$ supersymmetry. Therefore, the open strings stretched between two A-branes or between two B-branes are in $\mathcal{N} = 2$ representations. However, since these two kinds of branes preserve different $\mathcal{N} = 2$ symmetries, the strings stretched between A-branes and B-branes are not in $\mathcal{N} = 2$ representations and are only in $\mathcal{N} = 1$ representations.

The special point about this example is that since this model is a member of the $\mathcal{N} = 1$ discrete series, we know that there are no other branes which preserve $\mathcal{N} = 1$ supersymmetry. There might be, however, other branes which preserve only the Virasoro algebra and which satisfy all other consistency conditions.

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